MA 114 Worksheet #22: Parametric Curves

1. (a) How is a curve different from a parametrization of the curve?
   (b) Suppose a curve is parameterized by $(x(t), y(t))$ and that there is a time t_0 with $x'(t_0) = 0$, $x''(t_0) > 0$, and $y'(t_0) > 0$. What can you say about the curve near $(x(t_0), y(t_0))$?
   (c) What parametric equations represent the circle of radius 5 with center $(2, 4)$?
   (d) Represent the ellipse $\frac{x^2}{a^2} + \frac{y^2}{c^2} = 1$ with parametric equations.
   (e) Do the two sets of parametric equations

   $$y_1(t) = 5\sin(t),\ x_1(t) = 5\cos(t),\ 0 \le t \le 2\pi$$

   and

   $$y_2(t) = 5\sin(t),\ x_2(t) = 5\cos(t),\ 0 \le t \le 20\pi$$

   represent the same parametric curve? Discuss.

2. Consider the curve parametrized by $c(t) = (\sin(t) + \frac{t}{\pi}, \left(\frac{t}{\pi}\right)^2)$, for $0 \le t \le 2\pi$.
   (a) Plot the points given by $t = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{3\pi}{2}, 2\pi$.
   (b) Consider the derivatives of $x(t)$ and $y(t)$ when $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. What does this tell you about the curve near these points?
   (c) Use the above information to plot the curve.

3. Find a Cartesian equation for the following parametric curves. Sketch the curves to see if you solved them correctly.
   (a) $x = \sqrt{t}$, $y = 1 - t$ for $t \ge 0$.
   (b) $x = 3t - 5$, $y = 2t + 1$ for $t \in \mathbb{R}$.
   (c) $x = \cos(t)$, $y = \sin(t)$ for $t \in [0, 2\pi]$.

4. Represent each of the following curves as parametric equations traced just once on the indicated interval.
   (a) $y = x^3$ from $x = 0$ to $x = 2$.
   (b) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

5. A particle travels from the point $(2, 3)$ to $(-1, -1)$ along a straight line over the course of 5 seconds. Write down a set of parametric equations which describe the position of the particle for any time between 0 and 5 seconds.

6. For the following parametric curves, find an equation for the tangent to the curve at the specified value of the parameter.
   (a) $x = e^{\sqrt{t}}$, $y = t - \ln(t^2)$ at $t = 1$.
   (b) $x = \cos(\theta) + \sin(2\theta)$, $y = \cos(\theta)$, at $\theta = \pi/2$.

7. For the following parametric curve, find dy/dx.
   (a) $x = e^{\sqrt{t}}$, $y = t + e^{-t}$.
   (b) $x = t^3 - 12t$, $y = t^2 - 1$.
   (c) $x = 4\cos(t)$, $y = \sin(2t)$.

8. Find d^2y/dx^2 for the curve $x = 7 + t^2 + e^t$, $y = \cos(t) + \dfrac{1}{t}$, $0 < t \leq \pi$.

9. Find the arc length of the following curves.
   (a) $x = 1 + 3t^2$, $y = 4 + 2t^3$, $0 \leq t \leq 1$.
   (b) $x = 4\cos(t)$, $y = 4\sin(t)$, $0 \leq t \leq 2\pi$.
   (c) $x = 3t^2$, $y = 4t^3$, $1 \leq t \leq 3$.

10. Suppose you wrap a string around a circle. If you unwind the string from the circle while holding it taut, the end of the string traces out a curve called the *involute* of the circle. Suppose you have a circle of radius r centered at the origin, with the end of the string all the way wrapped up resting at the point $(r, 0)$. As you unwrap the string, define θ to be the angle formed by the x-axis and the line segment from the center of the circle to the point up to which you have unwrapped the string.
    (a) Draw a picture and label θ.
    (b) Show that the parametric equations of the involute are given by $x = r(\cos\theta + \theta\sin\theta)$, $y = r(\sin\theta - \theta\cos\theta)$. Hint: For any angle θ, the line segment between the point up to which you have unwrapped the string and the point at the end of the string has the same length as the arc of the circle corresponding to the angle θ.
    (c) Find the length of the involute for $0 \leq \theta \leq 2\pi$.

MA 114 MathExcel Worksheet # 22: Parametric Curves

1. Find the coordinates at times $t = 0, 2, 4$ of a particle following the path $x = 1 + t^3, y = 9 - 3t^2$.

2. Give two different parametrizations of the line through $(4, 1)$ with slope 2.

3. Express the following parametric equations in the form $y = f(x)$ by eliminating the parameter.

   (a) $x = t + 3, y = 4t$ for $t \in [0, 5]$

   (b) $x = t^{-1}, y = t^{-2}$ for $t \in [1, 2]$

   (c) $x = t, y = \arctan(t^3 + e^t)$ for $t \in \mathbb{R}$

4. Find parametrizations of the following curves satisfying the given conditions.

   (a) $y = 3x - 4, c(0) = (2, 2)$

   (b) $y = 3x - 4, c(3) = (2, 2)$