MA 114 Worksheet #14: Power Series

1. (a) Give the definition of the radius of convergence of a power series $\sum_{n=0}^{\infty} a_n x^n$

   (b) For what values of x does the series $\sum_{n=1}^{\infty} 2\big(\cos(x)\big)^{n-1}$ converge?

   (c) Find a formula for the coefficients c_k of the power series $\frac{1}{0!} + \frac{2}{1!}x + \frac{3}{2!}x^2 + \frac{4}{3!}x^3 + \cdots$.

   (d) Find a formula for the coefficients c_n of the power series $1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + x^6 + \cdots$.

   (e) Suppose $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$ where $c \neq 0$. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$.

   (f) Consider the function $f(x) = \frac{5}{1-x}$. Find a power series that is equal to $f(x)$ for every x satisfying $|x| < 1$.

   (g) Define the terms *power series*, *radius of convergence*, and *interval of convergence.*

2. Find the radius and interval of convergence for

   (a) $\sum_{n=0}^{\infty} \frac{(-1)^n n}{4^n}(x-3)^n$

   (b) $4\sum_{n=1}^{\infty} \frac{2^n}{n}(4x-8)^n$

   (c) $\sum_{n=0}^{\infty} \frac{x^{2n}}{(-3)^n}$

   (d) $\sum_{n=0}^{\infty} n!(x-2)^n$

   (e) $\sum_{n=0}^{\infty} (5x)^n$

   (f) $\sum_{n=0}^{\infty} \sqrt{n}x^n$

   (g) $\sum_{n=1}^{\infty} \frac{x^n}{\sqrt{n}}$

   (h) $\sum_{n=3}^{\infty} \frac{x^n}{3^n \ln n}$

   (i) $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^n}$

   (j) $\sum_{n=4}^{\infty} \frac{(-1)^n x^n}{n^4}$

   (k) $\sum_{n=3}^{\infty} \frac{(5x)^n}{n^3}$

3. Use term-by-term integration and the fact that $\int_0^x \frac{1}{1+t^2}\,dt = \arctan(x)$ to derive a power series centered at $x = 0$ for the arctangent function. HINT: $\frac{1}{1+x^2} = \frac{1}{1-(-x^2)}$.

4. Use the same idea as above to give a series expression for $\ln(1+x)$, given that $\int_0^x \frac{dt}{1+t} = \ln(1+x)$. You will again want to manipulate the fraction $\frac{1}{1+x} = \frac{1}{1-(-x)}$ as above.

5. Write $(1+x^2)^{-2}$ as a power series. HINT: use term-by-term differentiation.

MA 114 Math Excel Supplemental Worksheet #14: Power Series

1. Consider the series $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ and $g(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$.

   (a) Find the radius and interval of convergence of the series.

   (b) Find $f'(x), f''(x), f^{(3)}(x), f^{(4)}(x)$. Also, find some derivatives of $g(x)$ using term-by-term differentiation. What patterns do you notice between the derivatives of f and the derivatives of g?

   (c) What other functions can you think of that satisfy these properties?

2. Evaluate the following limits.

   (a) $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n$ where x is your favorite real number.

   (b) Use (a) to find $\lim_{n\to\infty} \left(1 - \frac{18}{2n}\right)^{2n}$

   (c) $\lim_{n\to\infty} n^{\frac{2}{n}}$

   (d) $\lim_{n\to\infty} \frac{2^{n+1} + (n+1)^2 - \log(2n+2)}{2^n + n^2 - \log(2n)}$

3. Use your knowledge of geometric series to find a power series representation of the following functions. For each power series that you find, write down its interval and radius of convergence.

   (a) $\frac{1}{8-x}$

   (b) $\frac{-1}{(1-x)^2}$

   (c) $\frac{22.1x}{x-x^6}$