Worksheet # 9: The Derivative, Velocities, and Tangent Lines

1. Comprehension check:
   (a) What is the definition of the derivative $f'(a)$ at a point a?
   (b) What is the geometric meaning of the derivative $f'(a)$ at a point a?
   (c) True or false: If $f(1) = g(1)$, then $f'(1) = g'(1)$.
   (d) True or false: If $f'(1) = g'(1)$, then $f(1) = g(1)$.

2. Part of Problem #1 on Worksheet #4 is given below. Rewrite each of these questions as a problem about the graph of $h(t)$, secant lines to the graph of $h(t)$, and/or tangent lines to the graph of $h(t)$.

   A ball is thrown vertically into the air from ground level with an initial velocity of 15 m/s. Its height at time t is $h(t) = 15t - 4.9t^2$.

   (a) How far does the ball travel during the time interval $[1, 3]$?
   (b) Compute the ball's average velocity over the time interval $[1, 3]$.
   (c) Compute the ball's average velocity over the time intervals $[1, 1.01]$, $[1, 1.001]$,$[0.99, 1]$, and $[0.999, 1]$.
   (d) Estimate the instantaneous velocity when $t = 1$.

3. (a) Find a function f and a number a so that the following limit represents a derivative $f'(a)$.

$$\lim_{h\to 0} \frac{(4+h)^3 - 64}{h}$$

   (b) Draw the graph of your function f from part (a) and the secant line whose slope is given by $\frac{(4+h)^3-64}{h}$.
   (c) Create a real-world scenario that is modeled by f, and write a problem about this scenario for which the answer is given by $\lim_{h\to 0} \frac{(4+h)^3 - 64}{h}$.

4. Let $f(x) = |x|$. Find $f'(1)$, $f'(0)$ and $f'(-1)$ or explain why the derivative does not exist.

5. The point $P = (3, 1)$ lies on the curve $y = \sqrt{x - 2}$.
   (a) If Q is the point $(x, \sqrt{x - 2})$, find a formula for the slope of the secant line PQ.
   (b) Using your formula from part (a) and a calculator, find the slope of the secant line PQ for the following values of x (do not round until you get to the final answer):

      2.9, 2.99, 2.999, 3.001, 3.01, and 3.1

      Tip: You can use you calculator by entering the formula under "y=" and then using "Table".
   (c) Using the results of part (b), guess the value of the slope of the tangent line to the curve at $P = (3, 1)$.
   (d) Verify that your guess is correct by computing an appropriate derivative.
   (e) Using the slope from part (d), find the equation of the tangent line to the curve at $P = (3, 1)$.

6. Let

$$g(t) = \begin{cases} at^2 + bt + c & \text{if } t \le 0 \\ t^2 + 1 & \text{if } t > 0 \end{cases}.$$

   Find all values of a, b, and c so that g is differentiable at $t = 0$.

7. Let $f(x) = e^x$.
   (a) Estimate the derivative $f'(0)$ by considering difference quotients $\frac{f(h)-f(0)}{h}$ for small values of h.
   (b) Compute the derivative $f'(0)$ exactly by finding $\lim_{h\to 0} \dfrac{f(h)-f(0)}{h}$.
8. Suppose that $f'(0)$ exists. Does the limit
$$\lim_{h\to 0} \frac{f(h)-f(-h)}{h}$$
exist? Can you express the limit in terms of $f'(0)$?
9. Find A and B so that the limit
$$\lim_{x\to 1} \frac{x^2+2x-(Ax+B)}{(x-2)^2}$$
is finite. Give the value of the limit.
10. Find the specified derivative for each of the following *using the limit definition of derivative.*
    (a) If $f(x) = 1/x$, find $f'(2)$.
    (b) If $g(x) = \sqrt{x}$, find $g'(2)$.
    (c) If $h(x) = x^2$, find $h'(s)$.
    (d) If $f(x) = x^3$, find $f'(-2)$.
    (e) If $g(x) = 1/(2-x)$, find $g'(t)$.

Supplemental Worksheet #9: Derivatives

1. Consider the constant function $f(x) = c$. Using the limit laws, prove that $f'(x) = 0$.
2. Given a differentiable function $f(x)$, define $F(x) = c \cdot f(x)$ for some real number c. Use the limit laws to show $F'(x) = c \cdot f'(x)$.
3. Define $F(x) = g(x) + h(x)$, where g and h are differentiable functions. Using the limit laws, show that $F'(x) = g'(x) + h'(x)$.
4. Use the previous three problems to find the derivative of $f(x) = 4x^2 - 10x + 2$ by decomposing this function into smaller pieces.
5. Find $g(2)$ and $g'(2)$ assuming the equation of the tangent line to $g(x)$ at $x = 3$ is given by $y = -10x + 20$.
6. Find a so that the tangent line to $f(x) = x^2$ at $x = a$ is perpendicular to the line $y = 7x + 3$.
7. The following limit gives the derivative for some function $f(x)$ at some value $x = a$. Find $f(x)$ and a:
$$\lim_{h\to 0} \frac{5^{2+h} + (2+h)^2 - 29}{h}$$
Find another function g and point b that will have the same derivative as above.
8. Suppose the function $f(x) = \frac{c^2}{x}$.
   (a) Use the definition of the derivative to find $f'(a)$ for a general point $x = a$ on the curve.
   (b) Use your result to find the equation of the tangent line at the point $(a, f(a))$.
   (c) Show that the area of the right triangle bounded by the tangent line, the x-axis, and the y-axis is always $2c^2$, no matter the value of a.

9. Let $f(x) = sin(x)$. Use the definition of derivative to find $f'(a)$ for a general point $x = a$. Hint: you will need the facts that:

$$sin(a+b) = sin(a)cos(b) + sin(b)cos(a)$$

$$\lim_{h\to 0} \frac{sin(h)}{h} = 1$$

$$\lim_{h\to 0} \frac{cos(h)-1}{h} = 0$$