Worksheet # 7: Intermediate Value Theorem and Limits at Infinity

Goals for MA 113 Recitations: This is a good time to remind ourselves about our goals for students in MA 113 recitations:

1. to develop your ability to make sense of problems and be persistent while solving them,
2. to develop your ability to productively collaborate with peers, and
3. to develop your ability to check your own work, i.e. to decide on your own whether or not your work is correct.

For each recitation, it is a good idea to have in mind one of these that you are going to actively think about while you work. Which will it be today?

1. State the Intermediate Value Theorem. Show $f(x) = x^3 + x - 1$ has a zero in the interval $(0, 1)$.
2. Using the Intermediate Value Theorem, find an interval of length 1 in which a solution to the equation $2x^3 + x = 5$ must exist.
3. Let $f(x) = \dfrac{e^x}{e^x - 2}$.
   (a) Show that $f(0) < 1 < f(\ln(4))$.
   (b) Can you use the Intermediate Value Theorem to conclude that there is a solution of $f(x) = 1$?
   (c) Can you find a solution to $f(x) = 1$?
4. (a) Show that the equation $xe^x = 2$ has a solution in the interval $(0, 1)$.
   (b) Determine if the solution lies in the interval $(0, 1/2)$ or $(1/2, 1)$.
   (c) Continue in this manner to find an interval of length $1/8$ which contains a solution of the equation $xe^x = 2$.
5. Consider the following piecewise function

$$f(x) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x > 0 \end{cases}$$

   Although $f(-1) = 0$ and $f(1) = 1$, $f(x) \neq 1/2$ for all x in its domain. Why doesn't this contradict the Intermediate Value Theorem?

6. Describe the behavior of the function $f(x)$ if $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to-\infty} f(x) = M$.
7. Explain the difference between "$\lim_{x\to-3} f(x) = \infty$" and "$\lim_{x\to\infty} f(x) = -3$".
8. Evaluate the following limits, or explain why the limit does not exist:
   (a) $\displaystyle\lim_{x\to\infty} \frac{3x^2 - 7x}{x - 8}$
   (b) $\displaystyle\lim_{x\to\infty} \frac{2x^2 - 6}{x^4 - 8x + 9}$
   (c) $\displaystyle\lim_{x\to-\infty} \frac{x}{x^6 - 4x^2}$
   (d) $\displaystyle\lim_{x\to-\infty} 3$
   (e) $\displaystyle\lim_{x\to\pm\infty} \frac{5x^3 - 7x^2 + 9}{x^2 - 8x^3 - 8999}$
   (f) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{x^{10} + 2x}}{x^5}$

9. Find the limits $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ if $f(x) = \left(\frac{x^2}{x+1} - \frac{x^2}{x-1}\right)$.

10. Sketch a graph with all of the following properties:

- $\lim_{t\to\infty} f(t) = 2$
- $\lim_{t\to-\infty} f(t) = 0$
- $\lim_{t\to 0^+} f(t) = \infty$
- $\lim_{t\to 0^-} f(t) = -\infty$
- $\lim_{t\to 4} f(t) = 3$
- $f(4) = 6$

Math Excel Worksheet #7 Supplemental Problems

1. Use the Intermediate Value Theorem to show that $\cos(x) = x$ has a solution in the interval $[0, 1]$.

2. Draw the graph of a function $f(x)$ on $[0, 4]$ with a jump discontinuity at $x = 2$ that still satisfies the conclusion of the Intermediate Value Theorem on $[0, 4]$, namely that "for every value M between $f(0)$ and $f(4)$, there exists at least one value $c \in (0, 4)$ such that $f(c) = M$."

3. Draw the graph of a function $g(x)$ on [1,6] with infinite one-sided limits at $x = 3$ that does not satisfy the conclusion of the Intermediate Value Theorem.

4. (Review) Evaluate the expression $\arcsin(\cos(\frac{\pi}{3}))$.

5. Find the instantaneous velocity of a particle with position given by $\sin(t)$ at $t = \pi/4$ by computing the following limit. (Hint: Use the substitution $t = \pi/4 + h$ and don't forget to change the limit accordingly!)

$$\lim_{t\to\frac{\pi}{4}} \frac{\sin(t) - \sin(\pi/4)}{t - \pi/4}.$$

You may find it useful to recall that $\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$.