Worksheet # 6: Limit Laws and Continuity

An Interesting Fact: Mathematicians did not use a formal theory of limits between the invention of calculus in the 1660's and the formal definition of a limit in the 1820's. Even after the 1820's, mathematicians and scientists wrote lim without writing $x \to a$ below it. It appears that the widespread use of $\lim\limits_{x\to a}$ was only adopted in the early 1900's after being used in several books, including one by G. H. Hardy titled "A Course of Pure Mathematics."

Remark on Notation: When working through a limit problem, your answers should be a chain of true equalities. Make sure to keep the $\lim\limits_{x\to a}$ operator until the very last step.

1. Given $\lim\limits_{x\to 2} f(x) = 5$ and $\lim\limits_{x\to 2} g(x) = 2$, use limit laws to compute the following limits or explain why we cannot find the limit.

   (a) $\lim\limits_{x\to 2} f(x)^2 + x \cdot g(x)^2$

   (b) $\lim\limits_{x\to 2} \dfrac{f(x)-5}{g(x)-2}$

   (c) $\lim\limits_{x\to 2} \dfrac{f(x)g(x)}{x}$

   (d) $\lim\limits_{x\to 2}(f(x)g(2))$

2. For each limit, evaluate the limit or or explain why it does not exist. Use the limit rules to justify each step. It is good practice to sketch a graph to check your answers.

   (a) $\lim\limits_{x\to 2} \dfrac{x+2}{x^2-4}$

   (b) $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2} - \dfrac{3}{x^2-x-2}\right)$

   (c) $\lim\limits_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$

   (d) $\lim\limits_{x\to 0} \dfrac{(2+x)^3-8}{x}$

3. Let $f(x) = 1 + x^2 \sin\left(\dfrac{1}{x}\right)$ for $x \neq 0$. Consider $\lim\limits_{x\to 0} f(x)$.

   (a) Find two simpler functions, g and h, that satisfy the hypothesis of the Squeeze Theorem.

   (b) Determine $\lim\limits_{x\to 0} f(x)$ using the Squeeze Theorem.

   (c) Use a calculator to produce a graph that illustrates this application of the Squeeze Theorem.

4. For each of the following tasks/problems, provide a specific example of a function $f(x)$ that supports your answer.

   (a) State the definition of continuity.

   (b) List the three things required to show f is continuous at a.

   (c) What does it mean for $f(x)$ to be continuous on the interval $[a, b]$? What does it mean to say only that "$f(x)$ is continuous"?

   (d) Identify the three possible types of discontinuity of a function at a point. Provide a sketch of each type.

5. Show that the following functions are continuous at the given point a using problem 4b.

   (a) $f(x) = \pi$, $a = 1$

   (b) $f(x) = \dfrac{x^2+3x+1}{x+3}$, $a = -1$

   (c) $f(x) = \sqrt{x^2-9}$, $a = 4$

6. Give the intervals of continuity for the following functions.

(a) $f(x) = \dfrac{x+1}{x^2+4x+3}$

(b) $f(x) = \dfrac{x}{x^2+1}$

(c) $f(x) = \sqrt{2x-3} + x^2$

(d) $f(x) = \begin{cases} x^2+1 & \text{if } x \leq 0 \\ x+1 & \text{if } 0 < x < 2 \\ -(x-2)^2 & \text{if } x \geq 2 \end{cases}$

7. Let c be a number and consider the function $f(x) = \begin{cases} cx^2 - 5 & \text{if } x < 1 \\ 10 & \text{if } x = 1 \\ \dfrac{1}{x} - 2c & \text{if } x > 1 \end{cases}$.

(a) Find all numbers c such that $\lim_{x \to 1} f(x)$ exists.

(b) Is there a number c such that $f(x)$ is continuous at $x = 1$? Justify your answer.

8. Find parameters a and b so that the following function is continuous

$$f(x) = \begin{cases} 2x^2 + 3x & \text{if } x \leq -4 \\ ax + b & \text{if } -4 < x < 3 \\ -x^3 + 4x^2 - 5 & \text{if } 3 \leq x \end{cases}$$

9. Suppose that

$$f(x) = \begin{cases} \frac{x-6}{|x-6|} & \text{if } x \neq 6, \\ 1 & \text{if } x = 6 \end{cases}$$

Determine the points at which the function $f(x)$ is discontinuous and state the type of discontinuity.

Math Excel Worksheet #6 Supplemental Problems

1. Let β be some fixed number greater than 0. Let ω and ϕ be constants and define a function $f(x)$ by the equation

$$f(x) := \begin{cases} x^2 - 2x + 3 & \text{if } x \leq 1 \\ \beta \cos(\omega x + \phi) & \text{if } x > 1 \end{cases}$$

Is it possible to find values for ω and ϕ such that $f(x)$ is continuous everywhere? If so, find those values. If not, prove that it is not possible. Does your answer depend on the value of β?

2. Find all values of c so that the following limits exist. Evaluate the corresponding limits.

(a) $\lim_{x \to c} \dfrac{2x^2 + 5x - 3}{x - c}$

(b) $\lim_{x \to 1} \dfrac{x^2 - 4x + c}{x + 1}$

3. Let $f(x) = x^2 + 1$.

(a) Give the intervals of continuity for $f(x)$.

(b) Find $g(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$.

(c) Evaluate $g(5)$.

(d) Sketch the graph of $f(x)$. Indicate how $g(5)$ is represented on the graph of $f(x)$.