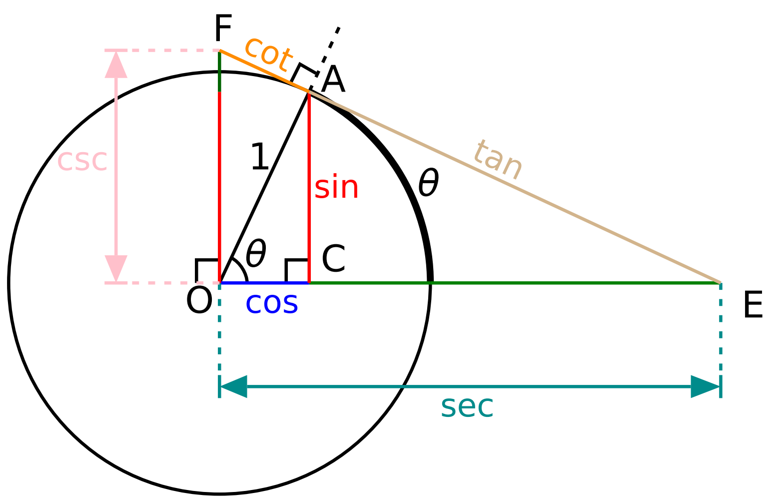
Worksheet # 3: Review of Trigonometry

An Interesting Fact: The study of trigonometry began in both ancient Greece and ancient India, and these two traditions were merged into roughly our modern form by Islamic mathematicians, notably Abu Nasr Mansur, around the year 1000. Islamic mathematicians established the connection between trig functions and the unit circle, and systematically studied the six fundamental trig functions.

> The etymology of the word 'sine' is instructive, for it shows what can happen as a result of imperfect linguistic and cultural filtering. The Sanskrit term for sine... was *jya-ardha* (half chord), which was later abbreviated to *jya*. From this came the phonetically derived Arabic word *jiba*... written as *jyb*. Early Latin translators, coming across this word, mistook it for anothe word, *jaib*... which was translated as *sinus*, which in Latin had a number of meanings... And hence the present word 'sine'. — George Gheverghese Joseph.



1. The key to understanding trig functions is to understand the unit circle — given an angle θ between 0 and $\pi/2$ (measured in radians!), each of the six trig functions measures a length related to the unit circle.

   (a) The word *radian* is an abbreviation of the phrase "radial angle." In a circle of radius r, one radian is defined to be the angle given by an arc of the circle having length r. Draw the unit circle in the plane and for each $m = 1, 2, 3, 4, 5, 6$, place a dot at the *approximate* point on the circle that is m radians counterclockwise from the point $(1, 0)$ (you won't be able to measure this precisely, just estimate it as best you can).

   (b) Recall that the definition of π is the ratio of the circumference to the diameter in a circle. Use your picture from the previous problem to explain why the value of π is greater than 3 but less than 3.5.

   (c) Define the functions $\sin(\theta)$ and $\cos(\theta)$ to be the lengths of the arcs AC and OC, respectively, on the diagram above. Explain why this definition of $\sin(\theta)$ and $\cos(\theta)$ agrees with the usual triangle-based definitions, i.e. that $\sin(\theta)$ is equal to "opposite" over "hypotenuse" in a right triangle.

   (d) Use different pairs of similar triangles to explain why each of the functions $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, and $\sec(\theta)$ measure the correspondingly labeled length in the picture above.

(e) Explain why $\sin^2(\theta)+\cos^2(\theta) = 1$ is equivalent to the Pythagorean theorem applied to the triangle OAC above, and why $1 + \tan^2(\theta) = \sec^2(\theta)$ is equivalent to the Pythagorean theorem applied to the triangle OAE above.

(f) For the unit circle, the radial angle of θ corresponds to an arc of length θ. Inverse trig functions are sometimes written $\arcsin(x)$ and $\arccos(x)$. Discuss with the other students in your group why it makes sense that the function $\sin(\theta)$ gives the length of the vertical line AC in the diagram above, while the function $\arcsin(x)$ is equal to the length of the circular arc for which the vertical line AC has length x; conclude that if $\sin(\theta) = x$, then $\arcsin(x) = \theta$. Discuss arccos similarly.

2. When θ is not between 0 and $\pi/2$, then we extend the definition of the trig functions as you have seen in your previous courses, allowing us to answer the following questions.

   (a) Suppose that $\sin(\theta) = 5/13$ and $\cos(\theta) = -12/13$. Find the values of $\tan(\theta)$, $\cot(\theta)$, $\csc(\theta)$, $\sec(\theta)$, and $\tan(2\theta)$.

   (b) If $\pi/2 \le \theta \le 3\pi/2$ and $\tan\theta = 4/3$, find $\sin\theta$, $\cos\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$.

   (c) Find all solutions of the equations (a) $\sin(x) = -\sqrt{3}/2$ and (b) $\tan(x) = 1$.

   (d) A ladder that is 6 meters long leans against a wall so that the bottom of the ladder is 2 meters from the base of the wall. Make a sketch illustrating the given information and answer the following questions: How high on the wall is the top of the ladder located? What angle does the top of the ladder form with the wall?

3. Let O be the center of a circle whose circumference is 48 centimeters. Let P and Q be two points on the circle that are endpoints of an arc that is 6 centimeters long. Find the angle between the segments OQ and OP. Express your answer in radians.

   Find the distance between P and Q.

4. Show that $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$.

5. Find the exact values of the following expressions. Do not use a calculator.

   (a) $\tan^{-1}(1)$

   (b) $\tan(\tan^{-1}(10))$

   (c) $\sin^{-1}(\sin(7\pi/3))$

   (d) $\tan(\sin^{-1}(0.8))$

6. Find all solutions to the following equations in the interval $[0, 2\pi]$. You will need to use some trigonometric identities.

   (a) $\sqrt{3}\cos(x) + 2\tan(x)\cos^2(x) = 0$

   (b) $3\cot^2(x) = 1$

   (c) $2\cos(x) + \sin(2x) = 0$

Supplemental Worksheet # 3: Trigonometric Functions

1. Use the addition formulas to compute $\cos(\frac{\pi}{3} - \frac{\pi}{4})$ exactly.
2. Derive the following identity using the basic identities from the text book.

$$\cos(2\theta) = 2\cos^2(\theta) - 1$$

3. Use the previous problem to show that $\cos(\frac{\pi}{8}) = \sqrt{\frac{1}{2} + \frac{\sqrt{2}}{4}}$.
4. Find $\cos(\frac{\pi}{16})$ exactly.
5. Let θ be the angle between the line $y = mx + b$ and the x-axis. Prove that $m = \tan\theta$.