Worksheet # 29: Review for Final

An Interesting Fact: Taylor polynomials were investigated in the 1600's in Europe. However, formulas equivalent to the Taylor polynomials for $\sin(x)$, $\cos(x)$, and $\arctan(x)$ had been previously discovered by Madhava of Sangamagrama in the Kerala region of India, two hundred years before the work of Taylor, Newton, Gregory, and others European mathematicians. Trigonometric functions were crucial ingredients for the study of astronomy, and many historians believe that the need to construct accurate astronomical tables led Madhava to develop this theory. While it is possible that Madhava's work was brought to European scholars by Jesuits traveling in India, thus influencing European mathematicians, there is not definitive evidence either for or against this transmission of knowledge to Europe.

1. Compute the following limits.

   (a) $\displaystyle\lim_{t\to 9} \frac{9-t}{3-\sqrt{t}}$

   (b) $\displaystyle\lim_{\theta\to 0} \frac{\sin(3\theta)}{10\theta}$

   (c) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2}$

   (d) $\displaystyle\lim_{x\to\infty} \frac{(\ln x)^2}{x}$

2. (a) State the limit definition of the continuity of a function f at $x = a$.

   (b) State the limit definition of the derivative of a function f at $x = a$.

   (c) Given $f(x) = \begin{cases} x^2 & \text{if } x < 1 \\ 4-3x & \text{if } x \geq 1 \end{cases}$. Is the function continuous at $x = 1$? Is the function differentiable at $x = 1$? Use the definition of the derivative. Graph the function to check your answer.

3. Provide the most general antiderivative of the following functions:

   (a) $f(x) = x^4 + x^2 + x + 1000$

   (b) $g(x) = (3x-2)^{20}$

   (c) $h(x) = \dfrac{\sin(\ln(x))}{x}$

4. Use implicit differentiation to find $\dfrac{dy}{dx}$, and compute the slope of the tangent line at (1,2) for the following curves:

   (a) $x^2 + xy + y^2 + 9x = 16$

   (b) $x^2 + 2xy - y^2 + x = 2$

5. An rock is thrown up the in the air and returns to the ground 4 seconds later. What is the initial velocity? What is the maximum height of the rock? Assume that the rock's motion is determined by the acceleration of gravity, 9.8 meters/second^2.

6. A conical tank with radius 5 meters and height 10 meters is being filled with water at a rate of 3 cubic meters per minute. How fast is the water level increasing when the water's depth is 3 meters?

7. A rectangular storage container with an open top is to have a volume of 10 cubic meters. The length of the container is twice its width. Material for the base costs \$10 per square meter while material for the sides costs \$6 per square meter. Find the cost of materials for the least expensive possible container.

8. (a) State the Mean Value Theorem.

   (b) If $3 \leq f'(x) \leq 5$ for all x, find the maximum possible value for $f(8) - f(2)$.

9. Use linearization to approximate $\cos(\frac{11\pi}{60})$

   (a) Write down $L(x)$ at an appropriate point $x = a$ for a suitable function $f(x)$.

   (b) Use part(a) to find an approximation for $\cos(\frac{11\pi}{60})$

   (c) Find the absolute error in your approximation.

10. Find the value(s) c such that $f(x)$ is continuous everywhere.

$$f(x) = \begin{cases} (cx)^3 & \text{if } x < 2 \\ \ln(x^c) & \text{if } x \geq 2 \end{cases}$$

11. (a) Find y' if $x^3 + y^3 = 6xy$.

    (b) Find the equation of the tangent line at $(3, 3)$.

12. Show that the function $f(x) = 3x^5 - 20x^3 + 60x$ has no absolute maximum or minimum.

13. Compute the following definite integrals:

    (a) $\displaystyle\int_{-1}^{1} e^{u+1}\,du$

    (b) $\displaystyle\int_{-2}^{2} \sqrt{4 - x^2}\,dx$

    (c) $\displaystyle\int_{1}^{9} \frac{x-1}{\sqrt{x}}\,dx$

    (d) $\displaystyle\int_{0}^{10} |x - 5|\,dx$

    Hint: For some of the integrals, you will need to interpret the integral as an area and use facts from geometry to compute the integral.

14. Write as a single integral in the form $\int_a^b f(x)\,dx$:

$$\int_{-2}^{2} f(x)\,dx + \int_{2}^{5} f(x)\,dx - \int_{-2}^{-1} f(x)\,dx$$

Math Excel Supplemental Problems # 29: Review for Final

1. State both parts of the Fundamental Theorem of Calculus.

2. Evaluate the following:

   (a) $\displaystyle\int_{0}^{4} (3x^{0.5} - 2xe^{-x^2})\,dx$

   (b) $\displaystyle\int_{0}^{1} \frac{e^{2x}}{1 + e^{2x}}\,dx$

   (c) $\displaystyle\int \frac{[\ln(s)]^2}{s}\,ds$

   (d) $\displaystyle\lim_{y\to 1} \frac{y^3 + 4y^2 - 3y - 2}{y^2 + 17y - 18}$

   (e) $\displaystyle\int (z^3 + 1)\sin(z^4 + 4z)\,dz$

   (f) $G'(2)$, if $G(x) = \displaystyle\int_{1}^{x^3} te^t\,dt$

   (g) $A'(x)$, if $A(x) = \displaystyle\int_{2}^{\sqrt{3x}} \sin(t)\,dt$

   (h) $\displaystyle\lim_{x\to 0} \frac{\tan(3x)}{5x}$

   (i) $\displaystyle\int_{-2}^{1} 3 + 2|x|\,dx$

3. A right cylinder with radius 2 and height 20, both measured in inches, is being filled with water. The water pours in at the rate of 10 cubic inches per second. Find the rate at which the level of the water is rising in the tank.

4. The base of a rectangle is on the x-axis and the other two corners lie above the x-axis on the curve given by $y = 6 - x^2$.
   (a) Sketch the curve $y = 6 - x^2$ and this rectangle.
   (b) Express the area of the rectangle as a function of a single variable.
   (c) Find the dimensions and area of the largest such rectangle. Justify your answer.

5. Assume that the derivative of a function satisfies $f'(x) = xe^{-x}$.
   (a) Find the intervals on which f is increasing, the intervals on which f is decreasing, and the local extrema of f.
   (b) Find the intervals on which f is concave down, the intervals on which f is concave up, and the points of inflection of f.

6. Use calculus to find the area of the triangle with the vertices $(2, 0), (0, 2)$, and $(-1, 1)$.

7. Evaluate the integral
$$\int_0^{2\pi} \sqrt{1 - \cos^2(x)}\, dx.$$

8. The cost in dollars of producing x units of a certain commodity is $C(x) = 7000 + 8x + 1/3x^2$. Find $C'(40)$ and describe what this number tells you in words. Estimate the cost of producing the 41^{st} unit.

9. Find $\int_0^2 f(x)\, dx$, where
$$f(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq 1 \\ x & \text{if } 1 < x \leq 2 \end{cases}.$$

10. Compute the derivative of the given functions:
    (a) $f(\theta) = \cos(2\theta^2 + \theta + 2)$
    (b) $g(u) = \ln\left(\sin^2(u)\right)$
    (c) $h(x) = \displaystyle\int_{-35915789}^{x} t^2 - te^{t^2+t+1}\, dt$
    (d) $r(y) = \arccos(y^3 + 1)$

11. Find an antiderivative for the function $f(x) = \dfrac{1+x}{1+x^2}$.

12. Give the interval(s) for which the function F is increasing. The function F is defined by
$$F(x) = \int_0^x \frac{5t - 3}{t^2 + 10}$$

13. Find a function $f(x)$ such that $f(e) = 0$ and $f'(x) = \dfrac{e^x - e^{-x}}{\ln x}$. (Hint: Consider the Fundamental Theorem of Calculus Part I)

14. Which of the following is an antiderivative for the function $f(x) = 2x\cos(x^2)$. Circle all the correct answers.

(a) $F(x) = -\sin(x^2)$

(b) $F(x) = \sin(x^2)$

(c) $F(x) = \int_0^{x^2} \sin(t)dt$

(d) $F(x) = \int_0^{x} 2t\cos(t^2)dt$

(e) $F(x) = \int_0^{x} 2t\sin(t^2)dt$

(f) $F(x) = \int_0^{x^2} \cos(t)dt$

15. Compute the indefinite integral $\int \frac{\sin(x)}{1+\cos^2(x)}dx$.