Worksheet # 27: Substitution and More Integration

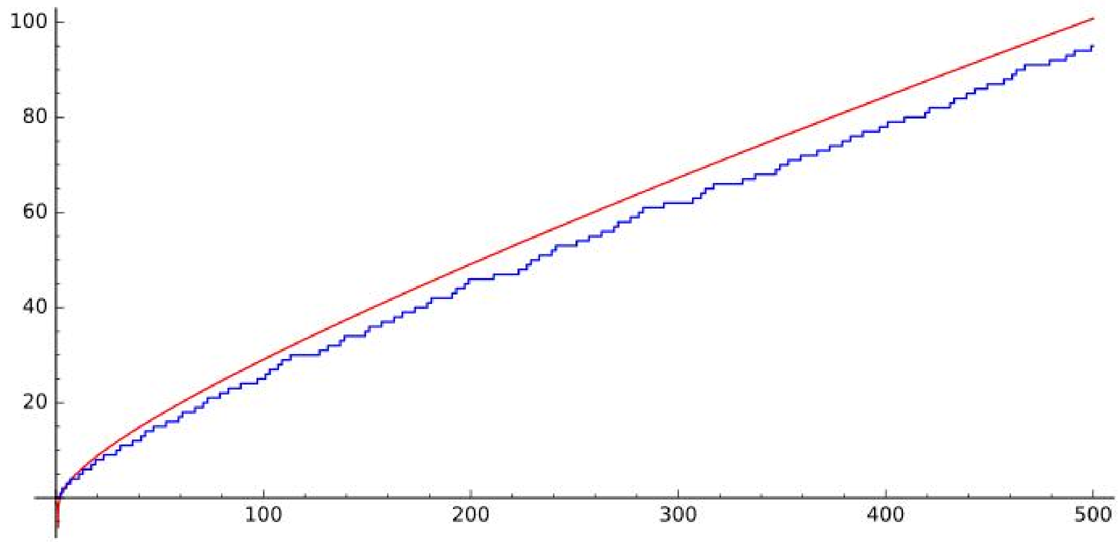

Figure 1: The graph of $\int_2^X \frac{1}{\log(t)}\,dt$ in red, and the graph of the number of primes less than X in blue, for X between 1 and 500.

An Interesting Fact: Many students ask "how can you do research in math?" because their sense is that all the questions in mathematics have been answered. However, this is not true at all! There are many (many!) unsolved problems in mathematics. Some of the most famous unsolved problems are called the Millennium Prize Problems — these are considered so important that the Clay Mathematics Institute has offered \$1,000,000 per problem in prize money for their solutions. Arguably the most famous of these is the *Riemann Hypothesis*, which claims that for a positive number X, the number of prime numbers less than X is roughly $\int_2^X \frac{1}{\log(t)}\,dt$ with what is known as "essentially square root accuracy". Unfortunately, we cannot use our substitution techniques to study this integral!

1. Evaluate the following indefinite integrals, and indicate any substitutions that you use:

   (a) $\int \frac{4}{(1+2x)^3}\,dx$

   (b) $\int x^2\sqrt{x^3+1}\,dx$

   (c) $\int \cos^4(\theta)\sin(\theta)\,d\theta$

   (d) $\int \sec^3(x)\tan(x)\,dx$

   (e) $\int e^x \sin(e^x)\,dx$

   (f) $\int \frac{2x+3}{x^2+3x}\,dx$

2. Evaluate the following definite integrals, and indicate any substitutions that you use:

   (a) $\int_0^7 \sqrt{4+3x}\,dx$

   (b) $\int_0^{\frac{\pi}{2}} \cos(x)\cos(\sin(x))\,dx$

   (c) $\int_0^4 \frac{x}{\sqrt{1+2x^2}}\,dx$

   (d) $\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

   (e) $\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$

3. (a) An oil storage tank ruptures and oil leaks from the tank at a rate of $r(t) = 100e^{-0.01t}$ liters per minute. How much oil leaks out during the first hour?
   (b) Is this model realistic? In other words, is it realistic to use this function $r(t)$ to model the leak rate in this situation? Why or why not?

4. If f is continuous on $(-\infty, \infty)$, prove that

$$\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(x)\,dx\,.$$

   For the case where $f(x) \geq 0$, draw a diagram to interpret this equation geometrically as an equality of areas.

5. Assume f is a continuous function.
   (a) If $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 x \cdot f(x^2)\,dx$.
   (b) If $\int_0^u f(x)\,dx = 1 + e^{u^2}$ for all real numbers u, find $\int_0^2 f(2x)\,dx$.

6. Do you remember our technique from worksheet #2 of writing $b^x = e^{x\ln(b)}$? Use this to find the indefinite integral $\int b^x\,dx$.

7. Which integral should be evaluated using substitution? Evaluate both integrals:
   (a) $\displaystyle\int \frac{9\,dx}{1+x^2}$
   (b) $\displaystyle\int \frac{x\,dx}{1+9x^2}$

8. Find a so that if $x = au$, then $\sqrt{16+x^2} = 4\sqrt{1+u^2}$.

9. Evaluate the following indefinite integrals, and indicate any substitutions that you use:
   (a) $\displaystyle\int \frac{dx}{x^2+3}$
   (b) $\displaystyle\int \frac{\cos(\ln(t))\,dt}{t}$
   (c) $\displaystyle\int \frac{x\,dx}{\sqrt{7-x^2}}$
   (d) $\displaystyle\int \frac{dt}{4t^2+9}$
   (e) $\displaystyle\int \frac{\ln(\arccos(x))\,dx}{\arccos(x)\sqrt{1-x^2}}$
   (f) $\displaystyle\int \frac{dt}{|t|\sqrt{12t^2-3}}$
   (g) $\displaystyle\int \frac{dx}{(4x-1)\ln(8x-2)}$
   (h) $\displaystyle\int e^{9-2x}\,dx$

MathExcel Worksheet # 27 Supplemental Problems

10. Evaluate the following:
    (a) $\displaystyle\int f(x)^3 f'(x)dx$
    (b) $\displaystyle\int \frac{f'(x)}{1+f(x)^2}dx$

11. An ellipse centered at the origin has equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that the area of this ellipse is given by πab by evaluating the appropriate integral. You will find it helpful to know the integral expression giving the area under the top half of the unit circle.