Worksheet # 23: Definite Integrals

The following summation formulas will be useful below.

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$$

1. Find the number n such that $\sum_{i=1}^{n} i = 78$.

2. Give the value of the following sums.

   (a) $\sum_{k=1}^{20}(2k^2+3)$

   (b) $\sum_{k=11}^{20}(3k+2)$

3. Recognize the sum as a Riemann sum and express the limit as an integral.

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{i^3}{n^4}$$

4. Let $f(x) = x$ and consider the partition $P = \{x_0, x_1, \ldots, x_n\}$ which divides the interval $[1, 3]$ into n subintervals of equal length.

   (a) Find a formula for x_k in terms of k and n.

   (b) We form a rectangle whose width is $\Delta x = (x_k - x_{k-1})$ and whose height is $f(x_k)$. Give the area of the rectangle.

   (c) Choose the sample points to be the right endpoint of each subinterval. Form the Riemann sum, and use the formula for sums of powers to simplify the Riemann sum.

   (d) Take the limit as n tends to infinity to find the area of the region under $f(x)$ for $1 \le x \le 3$.

   (e) Find the area above using geometry to check your answer.

5. Suppose $\int_0^1 f(x)\,dx = 2$, $\int_1^2 f(x)\,dx = 3$, $\int_0^1 g(x)\,dx = -1$, and $\int_0^2 g(x)\,dx = 4$.
   Compute the following using the properties of definite integrals:

   (a) $\int_1^2 g(x)\,dx$

   (b) $\int_0^2 [2f(x) - 3g(x)]\,dx$

   (c) $\int_1^1 g(x)\,dx$

   (d) $\int_1^2 f(x)\,dx + \int_2^0 g(x)\,dx$

   (e) $\int_0^2 f(x)\,dx + \int_2^1 g(x)\,dx$

6. Suppose that f is a continuous function and $6 \le f(x) \le 7$ for x in the interval $[3, 9]$.

   (a) Find the largest and smallest possible values for $\int_3^9 f(x)\,dx$.

   (b) Find the largest and smallest possible values for $\int_8^4 f(x)\,dx$.

7. Suppose that we know $\int_0^x f(t)\,dt = \sin(x)$, find the following integrals.

   (a) $\displaystyle\int_0^{\pi} f(t)\,dt$

   (b) $\displaystyle\int_{\pi/2}^{\pi} f(t)\,dt$

   (c) $\displaystyle\int_{-\pi}^{\pi} f(t)\,dt$

8. Find $\displaystyle\int_0^5 f(x)\,dx$ where $f(x) = \begin{cases} 3 & \text{if } x < 3 \\ x & \text{if } x \geq 3 \end{cases}$.

9. Simplify

$$\int_a^b f(t)\,dt + \int_b^c f(u)\,du + \int_c^a f(v)\,dv.$$

Math Excel Worksheet Supplementary Problems # 23

10. In this exercise, we evaluate the area A under the graph of $y = e^x$ over $[0, 1]$ using the formula for a geometric sum (valid for $r \neq 1$):

$$1 + r + r^2 + \cdots + r^{N-1} = \sum_{j=0}^{N-1} r^j = \frac{r^N - 1}{r - 1}$$

   (a) Show that $L_N = \dfrac{1}{N} \displaystyle\sum_{j=0}^{N-1} e^{j/N}$.

   (b) Apply the above formula for a geometric sum to prove $L_N = \dfrac{e-1}{N(e^{1/N}-1)}$.

   (c) Compute $A = \lim_{N\to\infty} L_N$ using L'Hôpital's Rule.

11. Use the result of Exercise 10(c) to show that the area under the graph of $f(x) = \ln x$ over $[1, e]$ is equal to 1. *Hint:* Relate the area under the graph of $f(x) = \ln x$ over $[1, e]$ to the area computed in Exercise 10.

12. Suppose that $\int_0^x f(t) = \cos(x)$ and $\int_0^x g(t) = 4x^2 - 7$. Compute the following

   (a) $\displaystyle\int_0^{\pi} (f(t) + g(t))\,dt$

   (b) $\displaystyle\int_{\pi}^{4\pi} f(t)\,dt$

   (c) $\displaystyle\int_3^{10} 2g(t)\,dt$

   (d) $\displaystyle\int_{-\pi}^{3\pi} (g(t) - f(t))\,dt$

13. Compute the integral $\displaystyle\int_0^4 (2x^2 + x)\,dx$ by computing Riemman sums for a partition of the interval into subintervals of equal length and then taking the limit as the number of subintervals approaches infinity.

14. Prove that for any function $f(x)$ on $[a, b]$,

$$R_N - L_N = \frac{b-a}{N}(f(b) - f(a)).$$