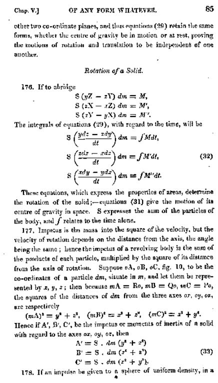
Worksheet # 22: Antiderivatives & Areas and Distances



Chap. V.] OF ANY FORM WHATEVER. 85

other two co-ordinate planes, and thus equations (29) retain the same forms, whether the centre of gravity be in motion or at rest, proving the motions of rotation and translation to be independent of one another.

Rotation of a Solid.

176. If to abridge

$$S\,(yZ - zY)\,dm = M,$$
$$S\,(zX - xZ)\,dm = M',$$
$$S\,(xY - yX)\,dm = M''.$$

The integrals of equations (29), with regard to the time, will be

$$S\left(\frac{ydz - zdy}{dt}\right)dm = \int M dt,$$
$$S\left(\frac{zdx - xdz}{dt}\right)dm = \int M' dt, \qquad (32)$$
$$S\left(\frac{xdy - ydx}{dt}\right)dm = \int M'' dt.$$

These equations, which express the properties of areas, determine the rotation of the solid;—equations (31) give the motion of its centre of gravity in space. S expresses the sum of the particles of the body, and $\int$ relates to the time alone.

177. Impetus is the mass into the square of the velocity, but the velocity of rotation depends on the distance from the axis, the angle being the same; hence the impetus of a revolving body is the sum of the products of each particle, multiplied by the square of its distance from the axis of rotation. Suppose oA, oB, oC, fig. 10, to be the co-ordinates of a particle dm, situate in m, and let them be represented by x, y, z; then because $mA = Ro$, $mB = Qo$, $mC = Po$, the squares of the distances of dm from the three axes ox, oy, oz, are respectively

$$(mA)^2 = y^2 + z^2, \quad (mB)^2 = x^2 + z^2, \quad (mC)^2 = x^2 + y^2.$$

Hence if A', B', C', be the impetus or moments of inertia of a solid with regard to the axes ox, oy, oz, then

$$A' = S\,.\,dm\,(y^2 + z^2)$$
$$B' = S\,.\,dm\,(x^2 + z^2) \qquad (33)$$
$$C' = S\,.\,dm\,(x^2 + y^2).$$

178. If an impulse be given to a sphere of uniform density, in a

An Interesting Fact: Antiderivatives, areas, and distances are fundamental in physics and mathematics. Pierre-Simon Laplace published *Mécanique Céleste* in five volumes between 1798 and 1825, and this is generally considered the next major work on gravitational mathematics and celestial mechanics after Newton's *Principia*. Eager to have a version in English, the Society for the Diffusion of Useful Knowledge commissioned a translated and expanded version from Mary Somerville, a famous Scottish mathematician and astronomer, which was published in 1831 under the title *The Mechanism of the Heavens*. In part due to the phenomenal success of her translation and extensions of the work of Laplace, in 1835 Somerville was one of the first two women (jointly with Caroline Herschel) to become a member of the Royal Astronomical Society.

1. Comprehension Check:

   (a) If F is an antiderivative of a continuous function f, is F continuous? What if f is not continuous?

   (b) Let $g(x) = \dfrac{x^3}{3} + 1$. Find $g'(x)$. Now give two antiderivatives of $g'(x)$.

   (c) Let $h(x) = x^2 + 1$, and let $H(x)$ be any antiderivative of h. What is $H'(x)$?

2. Find the most general antiderivative of the function $f(x) = x^2 - 3x + 2 - \frac{5}{x}$.

3. Find f given that
$$f'(x) = \sin(x) - \sec(x)\tan(x), \qquad f(\pi) = 1.$$

4. Find g given that
$$g''(t) = -9.8, \qquad g'(0) = 1, \qquad g(0) = 2.$$

   On the surface of the earth, the acceleration of an object due to gravity is approximately $-9.8\,\text{m/s}^2$. What situation could we describe using the function g? Be sure to specify what g and its first two derivatives represent.

5. The velocity of a train at several times is shown in the table below. Assume that the velocity changes linearly between each time given.

t=time in minutes	0	3	6	9
v(t)=velocity in Km/h	20	80	100	140

   (a) Plot the velocity of the train versus time.

(b) Compute the left and right-endpoint approximations to the area under the graph of v.

(c) Explain why these approximate areas are also an approximation to the distance that the train travels.

6. Let $f(x) = \frac{1}{x}$. Divide the interval $[1,3]$ into five subintervals of equal length and compute L_5 and R_5, the left and right endpoint approximations to the area under the graph of f in the interval $[1,3]$. Is R_5 larger or smaller than the true area? Is L_5 larger or smaller than the true area?

7. Let $f(x) = \sqrt{1-x^2}$. Divide the interval $[0,1]$ into four equal subintervals and compute L_4 and R_4, the left and right-endpoint approximations to the area under the graph of f. Is R_4 larger or smaller than the true area? Is L_4 larger or smaller than the true area? What can you conclude about the value π?

8. Write each of following in summation notation:

   (a) $1+2+3+4+5+6+7+8+9+10$

   (b) $2+4+6+8+10+12+14$

   (c) $2+4+8+16+32+64+128$.

9. Compute $\sum_{i=1}^{4}\left(\sum_{j=1}^{3}(i+j)\right)$.

10. Let $f(x) = x^2$.

    (a) If we divide the interval $[0,2]$ into n equal intervals of equal length, how long is each interval?

    (b) Write a sum which gives the right-endpoint approximation R_n to the the area under the graph of f on $[0,2]$.

    (c) Use one of the formulae for the sums of powers of k to find a closed form expression for R_n.

    (d) Take the limit of R_n as n tends to infinity to find an exact value for the area.

Math Excel Worksheet Supplementary Problems # 22:

11. Find constants c_1 and c_2 such that $F(x) = c_1xe^x + c_2e^x$ is an antiderivative of $f(x) = xe^x$.

12. Compute the following indefinite integrals (Don't forget the constant C):

    (a) $\int \frac{1-x^2}{x}\,dx$

    (b) $\int y^{2.6} - 4y + \frac{17}{y^{10}}, dy$

    (c) $\int 14s^{\frac{9}{5}}\,ds$

    (d) $\int 4\cos(\theta)\,d\theta$

13. Compute each of the following summations:

    (a) $\sum_{i=1}^{9}(2i-3)$

    (b) $\sum_{i=1}^{7}\frac{i^2-1}{3}$

    (c) $\sum_{j=1}^{6}2j^3$

    (d) $\sum_{i=1}^{5}\left(\sum_{j=1}^{6}(i^2-2j)\right)$

14. Let $g(x) = \sin^2(\theta)$. Divide the interval $[0, 2\pi]$ into 8 equal subintervals and compute L_8 and R_8, the left and right-endpoint approximations under the graph of g.

15. Let $h(x) = 2x^2 + x$.

    (a) If we divide the interval $[0, 4]$ into n equal subintervals, how long is each interval?

    (b) Write a sum which gives the right-endpoint approximation R_n to the area under the graph of h on $[0, 4]$.

    (c) Find a closed form for the expression R_n using summation formulae.

    (d) Take the limit of R_n as n tends to infinity to find an exact value for the area.