Worksheet # 20: L'Hôpital's Rule & Optimization

A N A L Y S E

D E S

INFINIMENT PETITS,

P O U R

L'INTELLIGENCE DES LIGNES COURBES.

Par M^r le Marquis DE L'HOSPITAL.

SECONDE EDITION.

A P A R I S,

Chez FRANÇOIS MONTALANT, Quay des Augustins.

M D C C X V.

AVEC APPROBATION ET PRIVILEGE DU ROY.

An Interesting Fact: L'Hôpital's Rule was probably not discovered by L'Hôpital!

> The Marquis de l'Hôpital was a French nobleman and amateur mathematician who [wanted to learn] calculus. [He] employed Johann Bernoulli to supply him with tracts on various aspects of the subject, as well as to provide him with any new mathematical discoveries of note. In a sense, it appears that l'Hôpital bought the rights to Bernoulli's mathematical research [and published it under his own name as] *Analyse de infiniment petits.* — William Dunham

1. Suppose we know:

$$\lim_{x \to a} f(x) = 0 \qquad \lim_{x \to a} g(x) = 0 \qquad \lim_{x \to a} p(x) = \infty \qquad \lim_{x \to a} q(x) = \infty$$

Which of the following limits are indeterminate forms? For those that are not an indeterminate form, evaluate the limit where possible.

(a) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$

(b) $\lim\limits_{x \to a} \dfrac{f(x)}{p(x)}$

(c) $\lim\limits_{x \to a} f(x)p(x)$

(d) $\lim\limits_{x \to a} \dfrac{p(x)}{f(x)}$

(e) $\lim\limits_{x \to a} p(x)q(x)$

(f) $\lim\limits_{x \to a} \dfrac{p(x)}{q(x)}$

2. Carefully state l'Hôpital's Rule.

3. Compute the following limits. Use l'Hôpital's Rule where appropriate, but first check that no easier method will solve the problem.

(a) $\lim\limits_{x \to 1} \dfrac{x^9 - 1}{x^5 - 1}$

(b) $\lim\limits_{x \to \infty} \dfrac{3x + 2\sqrt{x}}{1 - x}$

(c) $\lim\limits_{x \to 0} \dfrac{\sin(4x)}{\tan(5x)}$

(d) $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$

(e) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^3}$

(f) $\displaystyle\lim_{x\to-\infty} \frac{2x-5}{|3x+2|}$

(g) $\displaystyle\lim_{x\to\infty} \frac{5x^2+\sin x}{3x^2+\cos x}$

(h) $\displaystyle\lim_{x\to 1} \frac{x^2+2x-2}{x^2-2x+2}$

(i) $\displaystyle\lim_{x\to-\infty} x^2 e^x$

(j) $\displaystyle\lim_{x\to\infty} x^3 e^{-x^2}$

(k) $\displaystyle\lim_{x\to\pi} \frac{\cos(x)+1}{x^2-\pi^2}$

(l) $\displaystyle\lim_{x\to\infty} x\cdot\left(\arctan(x)-\frac{\pi}{2}\right)$

4. Find the value A for which we can use l'Hôpital's rule to evaluate the limit

$$\lim_{x\to 2} \frac{x^2+Ax-2}{x-2}.$$

For this value of A, give the value of the limit.

5. Find the dimensions of x and y of the rectangle of maximum area that can be formed using 3 meters of wire.

   (a) What is the constraint equation relating x and y?
   (b) Find a formula for the area in terms of x alone.
   (c) Solve the optimization problem.

6. Find two numbers whose difference is 100 and whose product is a minimum.

7. The sum of two positive numbers is 16. What is the smallest possible value of the sum of their squares?

8. A farmer wants to fence in an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

MathExcel Worksheet # 20 Supplemental Problems

9. Compute $\displaystyle\lim_{x\to 4} \frac{\sin(\pi\cdot x)}{x^2-16}$

10. Compute $\displaystyle\lim_{x\to\infty} x\cdot\sin(x/5)$

11. Compute $\displaystyle\lim_{x\to 0} \frac{\sin\left(x^2\right)}{x\cdot\tan(x)}$.

12. A box is to have a square base, no top, and a volume of 36 cubic centimeters. What are the dimensions of the box with the smallest possible total surface area? Provide an exact answer; do not convert your answer to decimal form. Make a sketch and introduce all the notation you are using.

13. Find the dimensions of the rectangle of largest area which can be inscribed in the closed region bounded by the x-axis, y-axis, and graph of $y = 8 - x^3$.

14. A sheet of cardboard 3 ft. by 4 ft. will be made into a box by cutting equal-sized squares from each corner and folding up the four edges. What will be the dimensions of the box with largest volume ?

15. A movie screen at a slumber party is 20 feet tall and is placed 10 feet above the floor. Jared decides to watch the movie while laying in his sleeping bag, so he can fall asleep partway through. The vertical viewing angle of a screen is the angle defined by the rays from the viewer to the top and bottom of the screen. At what distance x from the front of the room should Jared position himself so that the viewing angle is as large as possible ?