Worksheet # 2: The Exponential Function and the Logarithm

An Interesting Fact: The first book that gave a comprehensive discussion of both differential and integral calculus was written in 1748 by Maria Agnesi, an Italian philospher, theologian, humanitarian, and mathematician. This work was influential throughout Europe, and resulted in her election to the Bologna Academy of Sciences. Agnesi was the first woman appointed as a mathematics professor at a university.


INSTITUZIONI
ANALITICHE
AD USO
DELLA GIOVENTU' ITALIANA
DI D.NA MARIA GAETANA
AGNESI
MILANESE
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TOMO I.

IN MILANO, MDCCXLVIII.
NELLA REGIA-DUCAL CORTE.
CON LICENZA DE' SUPERIORI.

1. Many students find statements like $2^0 = 1$ and $2^{1/3} = \sqrt[3]{2}$ a bit mysterious, even though most of us have used them for years, so let's start there. Write down the list of numbers $2^1 = 2$, $2^2 = 2 \times 2 = 4$, $2^3 = 2 \times 2 \times 2 = 8$, thus
$$2^1, 2^2, 2^3, 2^4, 2^5, \ldots .$$
   (a) What do you multiply by to get from a number on this list to the next number to the right? Starting from any number *except* 2^1, what do you divide by to get from that number to the previous number on the left?

   (b) If we start at 2^1 and move to the left following this patten, it suggests how we should define 2^0. What do you get for 2^0 if you follow the pattern?

   (c) If we now move from 2^0 another number to the left following the pattern, it suggests how we should define 2^{-1}, and then 2^{-2}, etc. What do you get for these values if you follow the pattern?

   (d) Do these patterns help you make sense of the rule $2^{a+b} = 2^a \times 2^b$? Discuss this with the students in your group. (Bonus question: discuss whether or not these patterns and this rule help us make sense of the equations $2^{1/2} = \sqrt{2}$ and $2^{1/3} = \sqrt[3]{2}$.)

   (e) Does it matter for your reasoning that the base of the exponential was the number 2? Why or why not?

2. Find the solutions to the following computational problems by using properties of exponentials and logarithms.

   (a) Solve $10^{2x+1} = 100$.

   (b) Solve $2^{(x^2)} = 16$.

   (c) Solve $2^x = 4^{x+2}$.

   (d) Find $\log_2(8)$.

   (e) Find $\ln(e^2)$.

   (f) Solve $e^{3x} = 3$.

   (g) Solve $4^x = e$.

   (h) Solve $\ln(x+1) + \ln(x-1) = \ln(3)$. Be sure to check your answer.

3. (a) Graph the functions $f(x) = 2^x$ and $g(x) = 2^{-x}$ and give the domains and range of each function.
   (b) Determine if each function is one-to-one. Determine if each function is increasing or decreasing.
   (c) Graph the inverse function to f. Give the domain and range of the inverse function.

4. Since e^x and $\ln(x)$ are inverse functions, we can write $\heartsuit = e^{\ln(\heartsuit)}$ if the value of $\heartsuit$ is positive. This is super useful when dealing with exponential functions with complicated bases, because then you can use log laws to simplify the exponent on e. But, you have to be careful when you apply this rule, as the following examples show.
   (a) Explain why $b = e^{\ln(b)}$ is only true when $b > 0$. (Hint: think about the domain of natural log.)
   (b) Explain why for any $b > 0$, we have $b^a = e^{a\ln(b)}$.
   (c) Explain why $(\cos(x) + 3)^{\sin(x)+2} = e^{(\sin(x)+2)\ln(\cos(x)+3)}$ is true.
   (d) Explain why $\cos(x)^{\sin(x)} = e^{\sin(x)\ln(\cos(x))}$ is not true.
   (e) Let f be the function $f(x) = 4^x$. Find the value of k that allows you to write the function f in the form $f(x) = e^{kx}$.
   (f) Let f be the function $f(x) = 5 \cdot 3^x$. Find a k that allows you to write the function f in the form Ae^{kx}.

5. Evaluate the expressions $4^{(3^2)}$ and $(4^3)^2$. Are they equal?

6. Suppose a and b are positive real numbers and $\ln(ab) = 3$ and $\ln(ab^2) = 5$. Find $\ln(a)$, $\ln(b)$, and $\ln(a^3/\sqrt{b})$.

7. Suppose that a population doubles every two hours. If we have one hundred critters at 12 noon, how many will there be after 1 hour? after 2 hours? How many were there at 11am? Give a formula for the number of critters at t hours after 12 noon.

8. Suppose that f is a function of the form $f(x) = Ae^{kx}$. If $f(2) = 20$ and $f(5) = 10$, will we have $k > 0$ or $k < 0$? Find A and k so that $f(2) = 20$ and $f(5) = 10$.

9. The number e is mysterious and arises in many different ways.
   (a) Use your calculator to compute $(1 + \frac{1}{n})^n$ for n equal to $1, 2, 3, \ldots$. Compute this for larger and larger n until it does not make a difference in the decimal you get. What is the value you reach?
   (b) Use your calculator to compute the sums

$$\frac{1}{1} + \frac{1}{1 \cdot 2},$$
$$\frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3},$$
$$\frac{1}{1} + \frac{1}{1 \cdot 2} + \frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4},$$

   and so on, until it no longer makes a difference in the decimal you get. What is the value you reach?
   (c) The first two problems showed you two ways to approximate the value of e. The first way involved limits, which we will talk about soon in this course. The second involved "infinite series," which is a topic covered in Calculus II. As we will see later in this course, there is another way to define e — we will see that the area of the region enclosed by the line $x = 1$, the x-axis, the graph of $y = 1/x$, and the line $x = a$ is equal to $\ln(a)$. So, the value of a for which this area is 1 is $a = e$. Amazing!

Supplemental Worksheet # 2

1. Find real numbers a and b such that $\ln(ab) \neq \ln(a)\ln(b)$. Can you find values of a and b that make the statement false? Do you think the statement is usually true or false?

2. Determine whether f and g are the same function. Why or why not?

   (a) $f(x) = \ln(x^2)$, $g(x) = 2\ln(x)$

   (b) $f(x) = \sin^2(x) + \cos^2(x)$, $g(x) = 1$

   (c) $f(x) = \sec^2(x) - \tan^2(x)$, $g(x) = 1$

   (d) $f(x) = 5x^3$, $g(\beta) = 5\beta^3$

   We define the hyperbolic sine and cosine as follows:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}$$

3. Show that $\sinh(x)$ is odd and $\cosh(x)$ is even.

4. Compute $\sinh(\ln 5)$ without using a calculator.

5. (Challenge) Suppose that $\sinh(x) = 0.8$. Determine $\cosh(x)$. (Hint: Can you find an identity between $\cosh^2(x)$ and $\sinh^2(x)$?)