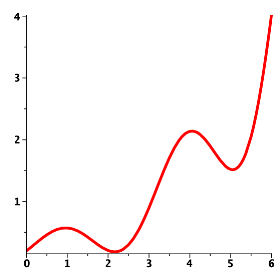
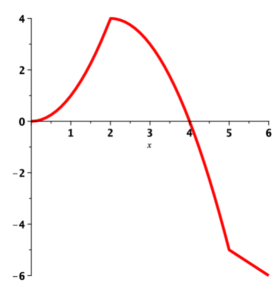
Worksheet # 19: The Shape of a Graph

1. Comprehension Check:
   (a) Explain what the First Derivative Test reveals about a continuous function $f(x)$ including when and how to use it.
   (b) Explain what the Second Derivative Test reveals about a twice differentiable function $f(x)$ and include how to use it. Does the test always work? What should you do if it fails?
   (c) Identify the similarities and differences between these two tests.

2. (a) Consider the function $f(x) = 2x^3 - 9x^2 - 24x + 5$ on $(-\infty, \infty)$.
      i. Find the critical number(s) of $f(x)$.
      ii. Find the intervals on which $f(x)$ is increasing or decreasing.
      iii. Find the local extrema of $f(x)$.
   (b) Repeat with the function $f(x) = \dfrac{x}{x^2+4}$ on $(-\infty, \infty)$.

3. Below are the graphs of two functions.





   (a) Find the intervals where each function is increasing and decreasing respectively.
   (b) Find the intervals of concavity for each function.
   (c) For each function, identify all local extrema and inflection points on the interval (0,6).

4. (a) Consider the function $f(x) = x^4 - 8x^3 + 5$.
      i. Find the intervals on which the graph of $f(x)$ is increasing or decreasing.
      ii. Find the inflection points of $f(x)$.
      iii. Find the intervals of concavity of $f(x)$.
   (b) Repeat with the function $f(x) = 2x + \sin(x)$ on $\left(-\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right)$.
   (c) Repeat with the function $f(x) = xe^x$.

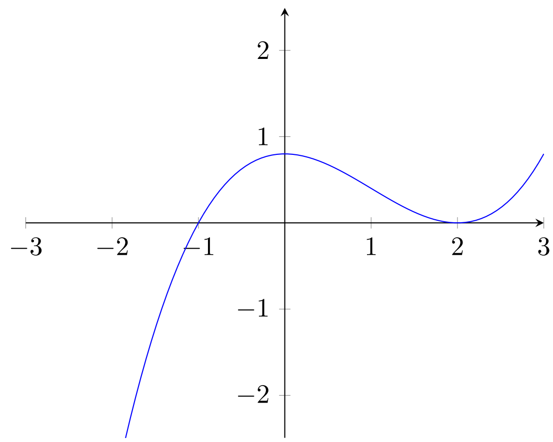
5. Find the local extrema of the following functions using the second derivative test (if possible):
   (a) $f(x) = x^5 - 5x + 4$
   (b) $g(x) = 5x - 10\ln(2x)$
   (c) $h(x) = 3x^5 - 5x^3 + 10$

6. Sketch a graph of a continuous function $f(x)$ with the following properties:
   - f is increasing on $(-\infty, -3) \cup (1, 7) \cup (7, \infty)$
   - f is decreasing on $(-3, 1)$

- f is concave up on $(0, 3) \cup (7, \infty)$
- f is concave down on $(-\infty, 0) \cup (3, 7)$

Math Excel Worksheet # 19: The Shape of a Graph

9. Consider the graph below.



   (a) Suppose the graph above is of the function $f(x)$. On which intervals is $f(x)$ increasing? Decreasing? Concave up? Concave down?

   (b) Instead, suppose that the graph above is of $f'(x)$. On which intervals is $f(x)$ increasing? Decreasing? Concave up? Concave down?

   (c) Finally, suppose the graph above is of $f''(x)$. On which intervals is $f(x)$ concave up? Concave down?

10. Sketch the graph of an increasing function $g(x)$ where $g''(x)$ changes from positive to negative at $x = 2$ and from negative to positive at $x = 4$. Do the same for a decreasing function.

11. Let $P(t) = te^{-t^2}$. Find the intervals where $P(t)$ is increasing and decreasing, all local extrama, and the intervals of concavity, and all inflection points.

12. (Review) For what values of a, m, and b does the function

$$f(x) = \begin{cases} 3 & \text{if } x = 0 \\ -x^2 + 3x + a & \text{if } 0 < x < 1 \\ mx + b & \text{if } 1 \le x \le 2 \end{cases}$$

satisfy the hypothesis of the Mean Value Theorem on the interval $[0, 2]$.