Worksheet # 18: Extreme Values and the Mean Value Theorem

Figure 1: An excerpt from *Hisab al-jabr w'al-muqabala*, courtesy of the Bodleian library.

An Interesting Fact: When we study extreme values of functions, we rely on our ability to do numerical calculations and solve algebraic equations. Our modern understanding of number systems and algebra can be traced to Baghdad in the 9th century, with the work of Muhammad ibn Musa al-Khwarizmi, who authored two incredibly influential books.

> [One of these books was] *Hisab al-jabr w'al-muqabala* (which may be loosely translated as Calculation by Reunion and Reduction)... In the twelfth century the book was translated into Latin under the title *Liber algebrae et almucabola*, thus giving a name to a central area of mathematics... [Another of al-Khwarizmi's books was] *Algorithmi de numero indorum*, which explained the Indian number system. While al-Khwarizmi was at pains to point out the Indian origin of this number system, subsequent translations of the book [were attributed to] the author. Hence, in Europe any scheme using these numerals came to be known as an "algorism", or, later, "algorithm" (a corruption of the name al-Khwarizmi). — George Gheverghese Joseph

1. Comprehension check:
   (a) True or False: If $f'(c) = 0$ then f has a local maximum or local minimum at c.
   (b) True or False: If f is differentiable and has a local maximum or minimum at $x = c$ then $f'(c) = 0$.
   (c) A function continuous on an open interval may not have an absolute minimum or absolute maximum on that interval. Give an example of continuous function on $(0, 1)$ which has no absolute maximum.
   (d) True or False: If f is differentiable on the open interval (a, b), continuous on the closed interval $[a, b]$, and $f'(x) \neq 0$ for all x in (a, b), then we have $f(a) \neq f(b)$.
2. Sketch the following:
   (a) The graph of a function defined on $(-\infty, \infty)$ with three local maxima, two local minima, and no absolute minima.
   (b) The graph of a continuous function with a local maximum at $x = 1$ but which is not differentiable at $x = 1$.
   (c) The graph of a function on $[-1, 1)$ which has a local maximum but not an absolute maximum.
   (d) The graph of a function on $[-1, 1]$ which has a local maximum but not an absolute maximum.
   (e) The graph of a discontinuous function defined on $[-1, 1]$ which has both an absolute minimum and absolute maximum.
3. State the definition of a critical number. Use this definition to find the critical numbers for the following functions:
   (a) $f(x) = x^4 + x^3 + 1$
   (b) $g(x) = e^{3x}(x^2 - 7)$

(c) $h(x) = |5x - 1|$

(d) $j(x) = (4 - x^2)^{1/3}$

4. Find the absolute maximum and absolute minimum values of the following functions on the given intervals. Specify the x-values where these extrema occur.

   (a) $f(x) = 2x^3 - 3x^2 - 12x + 1$, $[-2, 3]$

   (b) $h(x) = x + \sqrt{1 - x^2}$, $[-1, 1]$

   (c) $f(x) = 2\cos(x) + \sin(2x)$, $[0, \frac{\pi}{2}]$

   (d) $f(x) = x^{-2} \ln x$, $[\frac{1}{2}, 4]$

5. State the Mean Value Theorem. Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

   (a) $f(x) = \dfrac{x}{x+2}$ on the interval $[1, 4]$

   (b) $f(x) = \sin(x) - \cos(x)$ on the interval $[0, 2\pi]$

6. Use the Mean Value Theorem to show that $\sin(x) \leq x$ for $x \geq 0$. What can you say for $x \leq 0$?

7. Suppose that $g(x)$ is differentiable for all x and that $-5 \leq g'(x) \leq 3$ for all x. Assume also that $g(0) = 4$. Based on this information, use the Mean Value Theorem to determine the largest and smallest possible values for $g(2)$.

8. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 miles on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why did she deserve the ticket?

9. Suppose that we know that $1 \leq f(2) \leq 4$ and that $2 \leq f'(x) \leq 5$ for all x. What are the largest and smallest possible values for $f(6)$? What about $f(-1)$?

10. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$ on the interval $0 \leq x \leq 1$.

Math Excel Worksheet # 18: Extreme Values and the Mean Value Theorem

11. Use implicit differentiation to find all critical points of the curve $27x^2 = (x^2 + y^2)^3$. Be sure to give the x-value and the y-value of each point. Hint: There are 7 distinct critical points.

12. Does there exist any differentiable function f so that $f(0) = -1$, $f(2) = 4$, and $f'(x) \leq 2$ for all x? If so give an example of such a function. If not, explain why no such function can exist.

13. Determine which values of c satisfy the conclusion of the Mean Value Theorem for each function on the given interval.

    (a) $f(x) = 5x - 1; \quad [-10, 20]$

    (b) $f(x) = ax + b; \quad [p, q]$

14. Show that if $f(x)$ is any quadratic polynomial, then the midpoint $c = \frac{a+b}{2}$ satisfies the conclusion of the Mean Value Theorem on $[a, b]$ for any a and b.