Worksheet #17: Exponential Growth and Decay

An Interesting Fact: One of the amazing qualities of mathematics is that it has both artistic and scientific aspects, which is why we often refer to the difference between "pure mathematics" and "applied mathematics." Two important 20th-century figures in applied mathematics were Evelyn Boyd Granville and Mary G. Ross.

Evelyn Granville worked at IBM and the North American Aviation Space and Information Systems Division, as well as Cal State Los Angeles. She worked on the Apollo space program doing work in celestial mechanics, trajectory analysis, and computing. Granville was the second African-American woman to receive a Ph.D. in mathematics in a U.S. University, graduating from Yale in 1949.

Mary Ross was hired by Lockheed Aircraft as a mathematician during World War II. She went on to work for Lockheed's secret Skunk Works division, developing design concepts for space travel and working on the Agena rocket program. Ross was the first female Native American engineer. (Photos of Granville and Ross are subject to copyright, but can be found online.)

1. Solve the following equations for α:
   (a) $500 = 1000e^{20\alpha}$
   (b) $40 = \alpha e^{10k}$, where $k = \frac{\ln(2)}{7}$.
   (c) $100,000 = 40,000e^{0.06\alpha}$.
   (d) $\alpha = 2,000e^{36k}$, where $k = \frac{\ln(0.5)}{18}$.

2. Choose two of the previous four equations, write a problem which may be represented by the given equation. Be sure to explicitly state what each quantity represents, using correct units. Use a different type of scenario for each equation.

3. The mass of substance X decays exponentially. Let $m(t)$ denote the mass of substance X at time t where t is measured in hours. If we know $m(1) = 100$ grams and $m(10) = 50$ grams, find an expression for the mass at time t.

4. A certain cell culture grows at a rate proportional to the number of cells present. If the culture contains 500 cells initially and 800 after 24 hours, how many cells will there be after a further 12 hours?

5. Suppose that the rate of change of the mosquito population in a pond is directly proportional to the number of mosquitoes in the pond.
   $$\frac{dP}{dt} = KP$$
   where $P = P(t)$ is the number of mosquitoes at time t, t is measured in days and the constant of proportionality $K = .007$
   (a) Give the units of K.
   (b) If the population of mosquitoes at time $t = 0$ is $P(0) = 200$, how many mosquitoes will there be after 90 days?

6. Suppose that $P(t)$ gives the number of individuals in a population at time t where t is measured in years. Each year 23 out of 1000 individuals give birth and 11 out of 1000 individuals die.

   Find a differential equation of the form $P' = kP$ that the function P satisfies.

7. A lucky colony of rabbits is brought to a large island where there are no predators and unlimited food. Under these conditions, they will reproduce at such a rate that the population doubles every 9 years. After 3 years, a team of scientists determines that there are 7000 rabbits on the island.
   (a) How many rabbits were brought to the island originally?
   (b) How many rabbits will there be 12 years after their introduction to the island?
8. Suppose that f is a solution of the differential equation $f' = kf$, where k is a constant. Compute the derivative of $g(x) = e^{-kx}f(x)$ and show that g is constant. Explain why $f(x) = Ae^{kx}$.

Math Excel Supplemental Problems 17: Exponential Growth and Decay

(a) There was a chemical spill at the local rainforest, which has genetically modified the local kangaroo population to make them dangerously fertile. The rate of growth of the population is five times the number of kangaroos present at a given time.
   i. Let $P(t)$ be the population of kangaroos t years after the spill. Express $P'(t)$ in terms of $P(t)$. (Hint: Your answer should be a differential equation.)
   ii. Suppose the initial population is two kangaroos. Find the equation $P(t)$ that gives the population for all times t.
   iii. Find the year that the kangaroo population reaches one billion.

(b) The Gompertz differential equation

$$\frac{dP}{dt} = kP\ln(\frac{P}{M})$$

(where M and k are constants) was introduced in 1825 by the English mathematician Benjamin Gompertz and is still used today for modeling population growth and other applications. Unlike a logarithmic growth model, a Gompertz curve has an “S” shape, growing more slowly at the start and end of a time period. When used to model population growth, M indicates the carrying capacity of the population, and k is a growth constant.
   i. Show that $P(t) = Me^{ae^{kt}}$ satisfies the Gompertz differential equation for any constant a.
   ii. To model population growth in a population of 20 laboratory rats, a scientist assumes that the number $P(t)$ of rats alive at time t (in months) satisfies the Gompertz differential equation with $M = 200$ and $k = -0.15$ month^{-1}. Given that $P(0) = 20$, find $P(t)$ and determine the population after 1 year.
   iii. Find the limit of $P(t)$ as t approaches infinity.

(c) Let $P = P(t)$ be a quantity that obeys an exponential growth law with growth constant k. Show that P increases m-fold after an interval of $\frac{\ln(m)}{k}$ years.