Worksheet # 15: Related Rates

1. Let a and b denote the length in meters of the two legs of a right triangle. At time $t = 0$, $a = 20$ and $b = 20$. If a is decreasing at a constant rate of 2 meters per second and b is increasing at a constant rate of 3 meters per second. Find the rate of change of the area of the triangle at time $t = 5$ seconds.

2. A person 6 feet tall walks along a straight path at a rate of 4 feet per second away from a streetlight that is 15 feet above the ground. Assume that at time $t = 0$ the person is touching the streetlight.

   (a) Draw a picture to represent the situation.

   (b) Find an equation that relates the length of the person's shadow to the person's position (relative to the streetlight).

   (c) Find the rate of change in the length of the shadow when $t = 3$.

   (d) Find how fast is the tip of the person's shadow is moving when $t = 4$.

   (e) Does the precise time make a difference in these calculations?

3. A spherical snow ball is melting. The rate of change of the surface area of the snow ball is constant and equal to -7 square centimeters per hour. Find the rate of change of the radius of the snow ball when $r = 5$ centimeters.

4. The height of a cylinder is a linear function of its radius (i.e. $h = ar + b$ for some a, b constants). The height increases twice as fast as the radius r and $\frac{dr}{dt}$ is constant. At time $t = 1$ seconds the radius is $r = 1$ feet, the height is $h = 3$ feet and the rate of change of the volume is 16π cubic feet/second.

   (a) Find an equation to relate the height and radius of the cylinder.

   (b) Express the volume as a function of the radius.

   (c) Find the rate of change of the volume when the radius is 4 feet.

5. A water tank is shaped like a cone with the vertex pointing down. The height of the tank is 5 meters and diameter of the base is 2 meters. At time $t = 0$ the tank is full and starts to be emptied. After 3 minutes the height of the water is 4 meters and it is decreasing at a rate of 0.5 meters per minute. At this instant, find the rate of change of the volume of the water in the tank. What are the units for your answer? Recall that the volume of a right-circular cone whose base has radius r and of height h is given by $V = \frac{1}{3}\pi r^2 h$.

6. A plane flies at an altitude of 5000 meters and a speed of 360 kilometers per hour. The plane is flying in a straight line and passes directly over an observer.

   (a) Sketch a diagram that summarizes the information in the problem.

   (b) Find the angle of elevation 2 minutes after the plane passes over the observer.

   (c) Find rate of change of the angle of elevation 2 minutes after the plane passes over the observer.

7. A car moves at 50 miles per hour on a straight road. A house is 2 miles away from the road. What is the rate of change in the angle between the house and the car and the house and the road when the car passes the house.

8. A car moves along a road that is shaped like the parabola $y = x^2$. At what point on the parabola are the rates of change for the x and y coordinates equal?

9. Let $f(x) = \dfrac{1}{1+x^3}$ and $h(x) = \dfrac{1}{1+f(x)}$

   (a) Find $f'(x)$.

   (b) Use the previous result to find $h'(x)$.

   (c) Let $x = x(t)$ be a function of time t with $x(1) = 1$ and set $F(t) = h(x(t))$. If $F'(1) = 18$, find $x'(1)$.

Math Excel Worksheet #15: Related Rates

10 Alex and Jamie are in motorboats located at the center of a lake. At time $t = 0$, Alex begins traveling north at a speed of 35 mph. At the same time, Jamie takes off heading east at a speed of 25 mph.

   (a) How far apart are Alex and Jamie after traveling for 12 min?

   (b) At what rate are they separating after 12 min?

11 Assume that the top of a 5-meter ladder is sliding down a concrete wall. Let h denote the height of the top of the ladder from the ground at time t, and let x denote the distance from the wall to the ladder's bottom.

   a.) Assume the bottom slides away from the wall at a rate of 0.8 m/s. Find the velocity of the top of the ladder at $t = 2$ s if the bottom is initially 1.5 m from the wall.

   b.) Suppose that the top is sliding down the wall at a rate of 1.2 m/s. Calculate $\frac{dx}{dt}$ when $h = 3$m.

12 A particle travels along a curve $y = f(x)$ as in the figure below. Let $L(t)$ be the particle's distance from the origin, i.e. $L(t) = \sqrt{x^2 + f(x)^2}$ where x is a function of t.

   (a) Show that $\dfrac{dL}{dt} = \left(\dfrac{x + f(x)f'(x)}{\sqrt{x^2+f(x)^2}}\right)\dfrac{dx}{dt}$ if the particle's location at time t is $P = (x, f(x))$.

   (b) Calculate $L'(t)$ when $x = 1$ and $x = 2$ if $f(x) = \sqrt{3x^2 - 8x + 9}$ and $\frac{dx}{dt} = 4$.

   (c) Let θ be the angle in the figure, where $P = (x, f(x))$. Show that

$$\frac{d\theta}{dt} = \left(\frac{xf'(x) - f(x)}{x^2 + f(x)^2}\right)\frac{dx}{dt}.$$

   Hint: Differentiate $\tan\theta = \frac{f(x)}{x}$ and observe that $\cos\theta = \frac{x}{\sqrt{x^2+f(x)^2}}$.