Worksheet # 14: Derivatives of Logarithms and Rates of Change

MIRIFICI
Logarithmorum
Canonis deſcriptio,
Ejuſque uſus, in utraque
Trigonometria; ut etiam in
omni Logiſtica Mathematica,
Amplissimi, Facillimi, &
expeditiſſimi explicatio.
Authore ac Inventore,
IOANNE NEPERO,
Barone Merchiſtonii,
&c. Scoto.
EDINBURGI,
Ex officina ANDREÆ HART
Bibliopolæ, CIↃ. DC. XIV.

An Interesting Fact: While several people independently developed the idea of the logarithm, the most influential of these was John Napier through his 1614 book *Mirifici Logarithmorum Canonis Descriptio*. Napier developed the theory of logarithms to allow faster calculation, and it was so successful that within a few decades his logarithms had spread across the globe due to the promotion of Henry Briggs and Edward Wright in England, Bonaventura Cavalieri in Italy, Johannes Kepler in Germany, and Xue Fengzuo in China. Through reprintings of the book *Shu Li Ching Yün*, originally published in Beijing by Emperor K'ang Hsi, Napier's theory of logarithms reached Japanese mathematicians in the early 1700's.

1. Find the derivative of $f(x) = 3^x$. Compute the derivative of $\log_3(x)$. Explain the relationship between your answers.

2. Find the derivatives of the following functions.

   (a) $f(x) = \sqrt{x}\ln(x)$

   (b) $g(x) = \dfrac{\ln(x)}{1+\ln(x)}$

3. A particle moves along a line so that its position at time t is $p(t) = 3t^3 - 12t$ where $p(t)$ represents the distance to the right of the origin. Recall that *speed* is given by the absolute value of velocity.

   (a) Find the velocity and speed at time $t = 1$.

   (b) Find the acceleration at time $t = 1$.

   (c) Is the velocity increasing or decreasing when $t = 1$?

   (d) Is the speed increasing or decreasing when $t = 1$?

4. An object is thrown upward so that its height at time t seconds after being thrown is $h(t) = -4.9t^2 + 20t + 25$ meters. Give the position, velocity, and acceleration at time t.

5. An object is thrown upward with an initial velocity of 5 m/s from an initial height of 40 m. Find the velocity of the object when it hits the ground. Assume that the acceleration of gravity is 9.8 m/s^2.

6. An object is thrown upward so that it returns to the ground after 4 seconds. What is the initial velocity? Assume that the acceleration of gravity is 9.8 m/s^2.

7. Suppose that height of a triangle is equal to its base b. Find the instantaneous rate of change in the area respect to the base b when the base is 7.

8. Suppose that an object is shot into the air vertically with an initial velocity v_0 and initial height s_0, with acceleration due to gravity denoted by g. Let $s(t)$ denote the height of the object after t time units.
   (a) Explain why $s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$ (this is sometimes called "Galileo's formula").
   (b) If time is measured in seconds and distance in meters, what are the units for s_0, v_0, and g?
9. The cost in dollars of producing x bicycles is $C(x) = 4000 + 210x - x^2/1000$.
   (a) Explain why $C'(40)$ is a good approximation for the cost of the 41st bicycle.
   (b) How can you use the values of $C(40)$ and $C'(40)$ to approximate the cost of 42 bicycles?
   (c) Explain why the model for $C(x)$ is not a good model for cost. What happens if x is very large?
10. Suppose that a population of bacteria triples every hour and starts with 400 bacteria.
   (a) Find an expression for the number n of bacteria after t hours.
   (b) Use this expression to estimate the rate of growth of the bacteria population after 2.5 hours.
11. Think about the other science courses you are currently taking (or have taken in the past). Identify three to five examples of problems from those courses that involve computing rates of change where methods from calculus might be useful.

Math Excel Worksheet #14: Derivatives of Logarithms and Rates of Change

12 An equation of motion of the form $s = Ae^{-ct}\cos(\omega t + \delta)$ represents damped oscillation of an object. Find the velocity and acceleration of the object.

13 Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

   (a) Find $\frac{dF}{dr}$ and explain its meaning. What does the minus sign indicate?
   (b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of $2N/km$ when $r = 20,000$ km. How fast does this force change when $r = 10,000$ km?

14 If a tank holds 5000 gallons of water, which drains from the bottom of the tank in 40 minutes, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 5000(1 - \frac{1}{40}t)^2 \qquad 0 \leq t \leq 40$$

Find the rate at which water is draining from the tank after (a) 5 min, (b) 10 min, (c) 20 min, and (d) 40 min. At what time is the water flowing out the fastest? The slowest?