Worksheet # 13: Implicit Differentiation and Inverse Functions

1. Find the derivative of y with respect to x:
   (a) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = \pi^{\frac{2}{3}}$.
   (b) $e^y \sin(x) = x + xy$.
   (c) $\cos(xy) = 1 + \sin(y)$.

2. Consider the ellipse given by the equation $\dfrac{(x-2)^2}{25} + \dfrac{(y-3)^2}{81} = 1$.
   (a) Find the equation of the tangent line to the ellipse at the point (u, v) where $u = 4$ and $v > 0$.
   (b) Sketch the ellipse and the line to check your answer.

3. Find the derivative of $f(x) = \pi^{\tan^{-1}(\omega x)}$, where ω is a constant.

4. Let (a, b) be a point in the circle $x^2 + y^2 = 144$. Use implicit differentiation to find the slope of the tangent line to the circle at (a, b).

5. Let $f(x)$ be an invertible function such that $g(x) = f^{-1}(x)$, $f(3) = \sqrt{5}$ and $f'(3) = -\frac{1}{2}$. Using only this information find the equation of the tangent line to $g(x)$ at $x = \sqrt{5}$.

6. Let $y = f(x)$ be the unique function satisfying $\dfrac{1}{2x} + \dfrac{1}{3y} = 4$. Find the slope of the tangent line to $f(x)$ at the point $(\frac{1}{2}, \frac{1}{9})$.

7. The equation of the tangent line to $f(x)$ at the point $(2, f(2))$ is given by the equation $y = -3x + 9$. If $G(x) = \dfrac{x}{4f(x)}$, find $G'(2)$.

8. Differentiate both sides of the equation, $V = \dfrac{4}{3}\pi r^3$, with respect to V and find $\dfrac{dr}{dV}$ when $r = 8\sqrt{\pi}$.

9. Use implicit differentiation to find the derivative of $\arctan(x)$. Thus if $x = \tan(y)$, use implicit differentiation to compute dy/dx. Can you simplify to express dy/dx in terms of x?

10. (a) Compute $\dfrac{d}{dx} \arcsin(\cos(x))$.
    (b) Compute $\frac{d}{dx}(\arcsin(x) + \arccos(x))$. Give a geometric explanation as to why the answer is 0.
    (c) Compute $\dfrac{d}{dx}\left(\tan^{-1}\left(\dfrac{1}{x}\right) + \tan^{-1}(x)\right)$ and simplify to show that the derivative is 0. Give a geometric explanation of your result.

11. Consider the line through $(0, b)$ and $(2, 0)$. Let θ be the directed angle from the x-axis to this line so that $\theta > 0$ when $b < 0$. Find the derivative of θ with respect to b.

12. Let f be defined by $f(x) = e^{-x^2}$.
    (a) For which values of x is $f'(x) = 0$
    (b) For which values of x is $f''(x) = 0$

13. The notation $\tan^{-1}(x)$ is ambiguous. It is not clear if the exponent -1 indicates the reciprocal or the inverse function. If we allow both interpretations, how many different ways can you (correctly) compute the derivative $f'(x)$ for

$$f(x) = (\tan^{-1})^{-1}(x)?$$

    In order to avoid this ambiguity, we will generally use $\cot(x)$ for the reciprocal of $\tan(x)$ and $\arctan(x)$ for the inverse of the tangent function restricted to the domain $(-\pi/2, \pi/2)$.

MathExcel Worksheet # 13 Supplemental Problems

14. Use implicit differentiation to find $\frac{dy}{dx}$ and $\frac{dx}{dy}$ for each of the following functions:

    (a) $y\cos(x) + 3x = e^y(1+x) - y$

    (b) $y^2 - 3xy + x^2 = \tan(x) + y$

    (c) $x^5 + e^{y^2} - y\sin(1+x) = \cos(y+x) - xy^3$

    (d) $e^{y+x^2} - x - 4y^2\cot(x) = 2^{45}$

15. Use implicit differentiation to find the derivative of $\text{arccot}(x)$ by considering $x = \cot(y)$. (Hint: After completing the differentiation, you may want to draw a certain triangle.)