Worksheet # 12: Derivatives of Trigonometric Functions and the Chain Rule

Interesting Fact: The followers of Newton (in England) and Leibniz (in continental Europe) argued and fought regarding which of them should receive "credit" for inventing calculus. The reality is that both made important and independent contributions to the development of calculus. Émilie du Châtelet was a French mathematician and physicist who in 1749 completed her translation of Newton's *Principia* into French, expanding the influence of Newton's work in Europe — to this day, du Châtelet's translation is considered the definitive French translation of this work. One of the reasons this translation was so influential was that she did more than just translate. In her extensive commentary on the translation, du Châtelet made a crucial contribution to Newtonian mechanics via her conservation law for total energy. In addition to her translation of Newton, she also wrote an influential treatise on physics, *Institutions de Physique*, in 1740.

1. For each of these problems, explain why it is true or give an example showing it is false.
   (a) True or False: If $f'(\theta) = -\sin(\theta)$, then $f(\theta) = \cos(\theta)$.
   (b) True or False: If θ is one of the non-right angles in a right triangle and $\sin(\theta) = \frac{2}{3}$, then the hypotenuse of the triangle must have length 3.
2. Calculate the first five derivatives of $f(x) = \sin(x)$. Then determine $f^{(8)}$ and $f^{(37)}$
3. Let $f(t) = t + 2\cos(t)$.
   (a) Find all values of t where the tangent line to f at the point $(t, f(t))$ is horizontal.
   (b) What are the largest and smallest values for the slope of a tangent line to the graph of f?
4. (a) Carefully state the chain rule using complete sentences.
   (b) Suppose f and g are differentiable functions so that $f(2) = 3$, $f'(2) = -1$, $g(2) = \frac{1}{4}$, and $g'(2) = 2$. Find each of the following:
      i. $h'(2)$ where $h(x) = \sqrt{[f(x)]^2 + 7}$.
      ii. $l'(2)$ where $l(x) = f(x^3 \cdot g(x))$.
5. Differentiate both sides of the double-angle formula for the cosine function, $\cos(2x) = \cos^2(x) - \sin^2(x)$. Do you obtain a familiar identity?

6. Differentiate each of the following and simplify your answer.

   (a) $r(\theta) = \theta^3 \sin(\theta)$
   (b) $s(t) = \tan(t) + \csc(t)$
   (c) $h(x) = \sin(x)\csc(x)$
   (d) $g(x) = \sec(x) + \cot(x)$
   (e) $f(x) = \sqrt[3]{2x^3 + 7x + 3}$
   (f) $g(t) = \tan(\sin(t))$
   (g) $h(u) = \sec^2(u) + \tan^2(u)$
   (h) $f(x) = xe^{(3x^2+x)}$
   (i) $g(x) = \sin(\sin(\sin(x)))$

7. Given the following functions: $f(x) = \sec(x)$, and $g(x) = x^3 - 2x + 1$. Find:

   (a) $f(g(x))$
   (b) $f'(x)$
   (c) $g'(x)$
   (d) $f'(g(x))$
   (e) $(f \circ g)'(x)$

8. Find an equation of the tangent line to the curve at the given point.

   (a) $f(x) = x^2 e^{3x}$, $x = 2$
   (b) $f(x) = \sin(x) + \sin^2(x)$, $x = 0$

9. Compute the derivative of $\frac{x}{x^2+1}$ in two ways:

   (a) Using the quotient rule.
   (b) Rewrite the function $\frac{x}{x^2+1} = x(x^2+1)^{-1}$ and use the product and chain rule.

   Check that both answers give the same result.

10. Suppose that $k(x) = \sqrt{\sin^2(x) + 4}$. Find three functions f, g, and h so that $k(x) = f(g(h(x)))$.

11. Let $h(x) = f \circ g(x)$ and $k(x) = g \circ f(x)$ where some values of f and g are given by the table

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-1	4	4	-1	-1
2	3	4	3	-1
3	-1	-1	3	-1
4	3	2	2	-1

    Find: $h'(-1), h'(3)$ and $k'(2)$.

12. Find all x values so that $f(x) = 2\sin(x) + \sin^2(x)$ has a horizontal tangent at x.

13. Suppose that at the instant when the radius of a circle of a circle is 5 cm, the radius is decreasing at a rate of 3 cm/sec. Find the rate of change of the area of the circle when the the radius is 5 cm.

14. A particle's distance from the origin (in meters) along the x-axis is modeled by $p(t) = 2\sin(t) - \cos(t)$, where t is measured in seconds.

    (a) Determine the particle's speed (speed is defined as the absolute value of velocity) at π seconds.
    (b) Is the particle moving towards or away from the origin at π seconds? Explain.
    (c) Now, find the velocity of the particle at time $t = \dfrac{3\pi}{2}$. Is the particle moving toward the origin or away from the origin?
    (d) Is the particle increasing speed at $\frac{\pi}{2}$ seconds?

MathExcel Worksheet # 12 Supplemental Problems

15. Compute the first and second derivatives for each of the following functions:

    (a) $F(\theta) = \sin(\theta + \tan(\theta))$

    (b) $f(X) = e^{aX^2+bX+c}$

    (c) $\varphi(t) = \dfrac{\tan(t)}{\cos(t)+1}$

    (d) $T(\omega) = \dfrac{1}{\cot(\omega^2)}$

    (e) $h(y) = \tan(y)\cos(\sin(y))$

    (f) $G(s) = e^{\cos(s^2-1)}$

16. Is there a formula for the derivative of the composition of three function $\frac{d}{dx}[f(g(h(x))]$? How about for the composition of four functions $\frac{d}{dx}[f(g(h(k(x)))]$?