MathExcel Supplemental Worksheet I:
Graphs, L'Hôpital's Rule, and Optimization

1. Consider the function $f(x) = x^4(x-1)^3$.
   (a) Find the critical numbers of f.
   (b) What does the second derivative test tell you about the behavior of f at these critical points?
   (c) What does the first derivative test tell you?

2. Suppose $f(3) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
   (a) Sketch a possible graph for f.
   (b) How many possible solutions does the equation $f(x) = 0$ have? Why?
   (c) Is is possible that $f'(2) = \frac{1}{3}$? Why or why not?

3. Sketch the graph of a function that satisfies all of the following conditions:
   - $f'(x) > 0$ if $x \neq 2$, $f''(x) > 0$ if $x < 2$,
   - $f''(x) < 0$ if $x > 2$, f has inflection point at $(2, 5)$,
   - $\lim_{x\to\infty} f(x) = 8$, and $\lim_{x\to-\infty} f(x) = 0$.

4. Find a and b so that
$$\lim_{x\to 0} \frac{\sin(3x) + ax + bx^3}{x^3} = 0.$$

5. Compute $\displaystyle\lim_{x\to\infty} \frac{x^2 + 3x + 5}{8^x}$.

6. Compute $\displaystyle\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x} \right)$.

7. If an initial amount A_0 of money is invested at an interest rate r compounded n times a year, the value of the investment after t years is
$$A = A_0 \left(1 + \frac{r}{n}\right)^{nt}.$$
   If we let $n \to \infty$, we say that the interest is *compounded continuously.* Consider A as a continuous function of n. Use l'Hôpital's Rule to show that if interest is compounded continuously, then the value of the investment after t years is
$$A = A_0 e^{rt}.$$
   Hint: You may want to use the natural log to get the equation in a certain form.

8. (a) Show that
$$\lim_{x\to\infty} \frac{e^x}{x^n} = \infty$$
   for any positive integer n. This shows that the exponential function approaches infinity faster than any power of x.
   (b) Show that
$$\lim_{x\to\infty} \frac{\ln(x)}{x^p} = 0$$
   for any number $p > 0$. This shows that the logarithmic function approaches infinity more slowly than any power of x.

9. A right triangle has legs of length 5 and 12. A rectangle is inscribed inside this triangle with sides parallel to the legs of the triangle. What is the maximum area of such a rectangle?

10. Find the point (x, y) on the graph of $y = \sqrt{x}$ nearest to the point $(4, 0)$.

11. What angle θ between two edges of length 3 will result in an isosceles triangle with largest area?