Math Excel Worksheet C: Review for Exam I

1. (a) Solve the equation $4^{5x+1} = 9$.

   (b) Express the following quantity as a single logarithm:

   $$\log_3(x^2 - 9) - 4\log_3(x-1) + \frac{1}{2}\log_3(x)$$

   (c) Find the solutions of the equation

   $$\ln(x+10) - 2\ln(x-2) \;=\; 0.$$

2. Compute the following (possibly infinite) limits or show that they do not exist.

   (a) $\lim_{x\to\infty} (\sqrt{4x^2+1} - 2x)$

   (b) $\lim_{x\to\infty} \frac{7x^4 - 2x^2 + x - \pi}{3x^4 - x^3 + 2}$

   (c) $\lim_{x\to -2} \frac{2x^2 - 3}{x - 2}$

   (d) $\lim_{x\to \frac{3\pi}{2}} \frac{\sin(x)}{x}$

   (e) $\lim_{h\to 0} \frac{(h+4)^2 - 16}{h}$

   (f) $\lim_{x\to 1} \frac{3x+7}{x-1}$

   (g) $\lim_{x\to -3} \frac{\sin(x+3)}{x^2+5x+6}$

3. Evaluate the following limits using the limit laws:

   (a) $\lim_{x\to 1} \frac{x-4}{\sqrt{x}-2}$

   (b) $\lim_{x\to 0} \frac{\sin(7x)}{\sin(3x)}$

   (c) $\lim_{x\to -2} \frac{\ln(-5x)\cdot e^{-3x-6}}{x-2}$

4. Find a value of c such that $\lim_{x\to 2} \frac{x^2+3x+c}{x-2}$ exists. Is this value unique? What is the value of the limit?

5. James has found a function $f(x)$ satisfying $f(0) > 0$ and $f(5) > 0$ that is continuous on $[0,5]$. He claims the Intermediate Value Theorem implies that $f(x)$ does not have a zero in the interval $[0,5]$. Is he correct? Why?

6. Suppose that $f(x)$ is a continuous function with $f(0) \geq 0$ and $f(1) \leq 1$. Prove that f has a fixed point in $[0,1]$, i.e, there is at least one real number $x \in [0,1]$, such that $f(x) = x$. (*Hint:* Let $g(x) = f(x) - x$).

7. Use the Squeeze Theorem to evaluate the following limits:

   (a) $\lim_{x\to 0} \tan(x)\cos\left(\sin\left(\frac{1}{x}\right)\right)$

(b) $\lim_{x\to 0}\left(x^2 e^{\sin\left(\frac{1}{x}\right)} - 7\right)$

(c) $\lim_{\omega\to 0^+} \cos\left(3\omega + \frac{\pi}{2}\right)\arctan(\ln(\omega))$

8. Suppose a particle has position $f(x) = 5x^2 - 2x$ meters at time x seconds.
   (a) Find a formula for the average velocity of the particle over the time interval $[2, 2+t]$.
   (b) Estimate the instantaneous velocity of the particle at time 2 seconds using the following three values for t: $-0.1, 0.1, 0.01$.

9. What is the average rate of change of the function $f(x) = x^2 + 2x - 3$ over the interval [1,4]? What is the instantaneous rate of change of $f(x)$ at $x = 2$?

10. Suppose that c is a constant and let

$$f(x) = \begin{cases} cx + 2 & x < 1 \\ x^2 + 3 & x > 1 \end{cases}$$

   (a) Determine the right and left-hand limits,

$$\lim_{x\to 1^+} f(x) \text{ and } \lim_{x\to 1^-} f(x).$$

   (b) Find the value of c so that $\lim_{x\to 1} f(x)$ exists.
   (c) For the value of c you found in (b), is the function f continuous at $x = 1$? Justify your answer.

11. Find the values of c and d which make the following function continuous everywhere:

$$f(x) = \begin{cases} -2x + 7, & x \le 2 \\ cx + d, & 2 < x \le 5 \\ x^2 - 13, & 5 < x \end{cases}$$

12. Evaluate the following:
   (a) $\sin(\arcsin(1))$
   (b) $\arctan(-\frac{1}{\sqrt{3}})$
   (c) $\arccos(\cos(\frac{5\pi}{3}))$
   (d) $\arcsin(\cos(\frac{7\pi}{6}))$

13. Consider the function

$$f(x) = \frac{2x^2 - 4x + 3}{x^2 - x - 6}.$$

   (a) Find the one-sided limits $\lim_{x\to 3^+} f(x)$ and $\lim_{x\to 3^-} f(x)$
   (b) Does $\lim_{x\to 3}$ exist?
   (c) Is $f(x)$ continuous at $x = 3$?

14. Show that the function

$$f(x) = \begin{cases} x^4 sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

   is continuous on $(-\infty, \infty)$.

15. (a) Show that the absolute value function $F(x) = \mid x \mid$ is continuous everywhere.
   (b) Prove that if f is continuous on an interval $[a, b]$, then so is $\mid f \mid$.
   (c) Is the converse of (b) true? That is, does $\mid f \mid$ being continuous on $[a, b]$ imply that f is also continuous on $[a, b]$? Justify your answer.